Fibred Path Categories

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1 Path Categories

We review the theory of path categories [1].

Definition 1. A *path category* is a category $\mathcal{C}$ equipped with two classes $\mathscr{F}$ and $\mathscr{W}$ of arrows in $\mathcal{C}$ satisfying the following axioms. Here we call an arrow in $\mathscr{F}$ a *fibration* and an arrow in $\mathscr{W}$ a *weak equivalence*. An arrow that is both a fibration and weak equivalence is called an *acyclic fibration*.

1 $\mathcal{C}$ has a terminal object 1 and every arrow $A \to 1$ to the terminal object is a fibration.

2 Fibrations are closed under composition.

3 The pullback of a fibration along any arrow exists and is again a fibration.

4 The pullback of an acyclic fibration along any arrow exists and is again an acyclic fibration.

5 Weak equivalences satisfy 2-out-of-6: if $f : A \to B$, $g : B \to C$ and $h : C \to D$ are composable arrows and both gf and hg are weak equivalences, then so are f, g, h and hgf.

6 Isomorphisms are acyclic fibrations and every acyclic fibration has a section.

7 For any object A, there exists a *path object* $\mathrm{P}A$, that is, an object $\mathrm{P}A$ equipped with a weak equivalence $\mathsf{r} : A \to \mathrm{P}A$ and a fibration $(\mathsf{s}, \mathsf{t}) : \mathrm{P}A \to A \times A$ (the product $A \times A$ exists by Axioms 1 and 3) such that the composite $(\mathsf{s}, \mathsf{t})\mathsf{r}$ is the diagonal $A \to A \times A$.

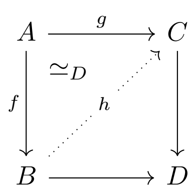
It is known that Axiom 7 of Definition 1 is, under the other assumptions, equivalent to the following stronger condition.

7′ For any arrow $f : A \to B$, there exist an object C, a weak equivalence $g : A \to C$ and a fibration $h : C \to B$ such that $hg = f$.

In particular, for a fibration $A \to C$, we have a *path object* $\mathrm{P}_C A$ *of* A *over* C, that is, an object $\mathrm{P}_C A$ equipped with a weak equivalence $\mathsf{r} : A \to \mathrm{P}_C A$ and a fibration $(\mathsf{s}, \mathsf{t}) : \mathrm{P}_C A \to A \times_C A$ (the fibre product $A \times_C A$ exists by Axiom 3) such that the composite $(\mathsf{s}, \mathsf{t})\mathsf{r}$ is the diagonal $A \to A \times_C A$.

Definition 2. Let $\mathcal{C}$ be a path category. Let $A \to C$ be an arrow, $B \to C$ a fibration and $f, g : A \to B$ arrows over C. A *homotopy from* f *to* g *over* C is an arrow $h : A \to \mathrm{P}_C B$ into some path object of B over C such that $\mathsf{s}h = f$ and $\mathsf{t}h = g$. We write $f \simeq_C g$ when there exists a homotopy from f to g over C.

Theorem 3 (Van den Berg and Moerdijk [1, Theorem 2.38])**.** *Let* $\mathcal{C}$ *be a path category,* $D \in \mathcal{C}$ *an object,* $A \to D$ *and* $B \to D$ *arrows,* $C \to D$ *a fibration, and* $f : A \to B$ *and* $g : A \to C$ *arrows over* D*. If* f *is a weak equivalence, then there exists an arrow* $h : B \to C$ *over* D *such that* $hf \simeq_D g$*.*



Definition 4. We say a functor $F : \mathcal{C} \to \mathcal{D}$ between path categories is *exact* if it preserves terminal objects, fibrations, weak equivalences and pullbacks of fibrations.

Definition 5. We say an exact functor $F : \mathcal{C} \to \mathcal{D}$ between path categories is a *weak equivalence* if it induces an equivalence between the homotopy categories.

2 Fibred Path Categories

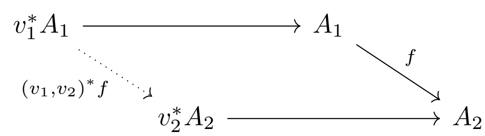
Definition 6. Let $\mathcal{B}$ be a path category. A *fibred path category over* $\mathcal{B}$ is a category $\mathcal{E}$ equipped with a functor $Q : \mathcal{E} \to \mathcal{B}$ and two classes $\mathscr{F}$ and $\mathscr{W}$ of arrows of $\mathcal{E}$ satisfying the following axioms. Here we call an arrow in $\mathscr{F}$ a *fibration*, an arrow in $\mathscr{W}$ a *weak equivalence* and an arrow in $\mathscr{F} \cap \mathscr{W}$ an *acyclic fibration.*

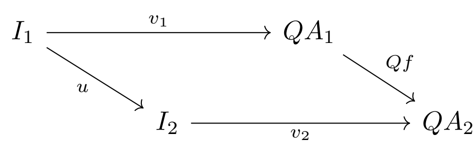
1 Q is a Grothendieck fibration.

2 Q sends fibrations and weak equivalences in $\mathcal{E}$ to fibrations and weak equivalences, respectively, in $\mathcal{B}$.

3 Fibrations and weak equivalences in $\mathcal{E}$ are stable under reindexing: for an arrow $f : A_1 \to A_2$ in $\mathcal{E}$, an arrow $u : I_1 \to I_2$ in $\mathcal{B}$ and a square $(v_1, v_2) : u \to Qf$, if f is a fibration (weak equivalence) in $\mathcal{E}$ and u is a fibration (weak equivalence) in $\mathcal{B}$, then the induced arrow $(v_1, v_2)^* f : v_1^* A_1 \to v_2^* A_2$

in $\mathcal{E}$ over u is a fibration (weak equivalence).





4 $\mathcal{E}$ has a terminal object 1 preserved by Q and every arrow $A \to 1$ to the terminal object in $\mathcal{E}$ is a fibration.

5 Fibrations in $\mathcal{E}$ are closed under composition.

6 The pullback of a fibration in $\mathcal{E}$ along any arrow exists and is again a fibration and preserved by Q.

7 The pullback of an acyclic fibration in $\mathcal{E}$ along any arrow exists and is again an acyclic fibration and preserved by Q.

8 Weak equivalences in $\mathcal{E}$ satisfy 2-out-of-6.

9 Isomorphisms are acyclic fibrations and every acyclic fibration has the *section lifting property*: for any acyclic fibration $f : A \to B$ in $\mathcal{E}$ and section $v : QB \to QA$ of Qf, there exists a section $g : B \to A$ of f over v.

10 For any object A in $\mathcal{E}$ and any path object $(I, \bar{\mathsf{r}}, \bar{\mathsf{s}}, \bar{\mathsf{t}})$ of QA, there exists a *path object* $\mathrm{P}_I A$ *of* A *over* I, that is, an object $\mathrm{P}_I A \in \mathcal{E}$ over I equipped with a weak equivalence $\mathsf{r} : A \to \mathrm{P}_I A$ over $\bar{\mathsf{r}}$ and a fibration $(\mathsf{s}, \mathsf{t}) : \mathrm{P}_I A \to A \times A$ over $(\bar{\mathsf{s}}, \bar{\mathsf{t}})$ (the product $A \times A$ exists and is over $QA \times QA$ by Axioms 4 and 6) such that the composite $(\mathsf{s}, \mathsf{t})\mathsf{r}$ is the diagonal $A \to A \times A$.

Axioms 9 and 10 of Definition 6 are, under the other assumptions, equivalent to the following stronger conditions.

9′ Isomorphisms are acyclic fibrations and, for any arrow $A \to C$ and acyclic fibration $B \to C$ in $\mathcal{E}$ and arrow $u : QA \to QB$ over QC, there exists an arrow $f : A \to B$ over C and over u.

10′ For any arrow $f : A \to B$ in $\mathcal{E}$, object $K \in \mathcal{B}$, weak equivalence $v : QA \to K$ and fibration $w : K \to QB$ such that $wv = Qf$, there exist an object $C \in \mathcal{E}$ over K, a weak equivalence $g : A \to C$ over v and a fibration $h : C \to B$ over w such that $hg = f$.

Remark 7. The stability under reindexing (Axiom 3 of Definition 6) contains some important cases:

- a cartesian arrow over a fibration (weak equivalence) is a fibration (weak equivalence) (when $f = 1$ and $v_2 = 1$). By a *horizontal fibration (weak equivalence)* we mean a cartesian arrow over a fibration (weak equivalence);

- every fibration (weak equivalence) in $\mathcal{E}$ factors as a vertical fibration (weak equivalence) followed by a horizontal fibration (weak equivalence) (when $u = 1$ and $v_1 = 1$);

- the reindexing along an arbitrary arrow preserves vertical fibrations (weak equivalences) (when $u = 1$ and $Qf = 1$).

One can show that these three conditions are equivalent to Axiom 3 under the assumption that isomorphisms are fibrations (weak equivalences) and fibrations (weak equivalences) are closed under composition.

Example 8. Let $\mathcal{C}$ be a path category. We denote by $(\mathcal{C}^{\to})_{\mathrm{f}}$ the full subcategory of $\mathcal{C}^{\to}$ consisting of fibrations. The category $(\mathcal{C}^{\to})_{\mathrm{f}}$, together with the codomain functor $(\mathcal{C}^{\to})_{\mathrm{f}} \to \mathcal{C}$, is a fibred path category over $\mathcal{C}$ with the Reedy fibrations and the levelwise weak equivalences.

Definition 9. Let $\mathcal{E}$ and $\mathcal{F}$ be fibred path categories over a path category $\mathcal{B}$. A *fibred exact functor $\mathcal{E} \to \mathcal{F}$ over $\mathcal{B}$* is a fibred functor $S : \mathcal{E} \to \mathcal{F}$ over $\mathcal{B}$ that preserves terminal objects, fibrations, weak equivalences and pullbacks of fibrations.

2.1 The Total Path Category

Theorem 10. *Let $Q : \mathcal{E} \to \mathcal{B}$ be a Grothendieck fibration and suppose the following:*

- *$\mathcal{B}$ is a path category;*

- *$\mathcal{E}$ is equipped with a class of fibrations and a class of weak equivalences;*

- *fibrations and weak equivalences in $\mathcal{E}$ are stable under reindexing.*

Then $\mathcal{E}$, together with Q, is a fibred path category over $\mathcal{B}$ if and only if $\mathcal{E}$ is a path category and Q is an exact functor.

Proof. The "only if" part is immediate from the definition. To show the "if" part, suppose that $\mathcal{E}$ is a path category and Q is an exact functor. We have to check that $\mathcal{E}$ satisfies Axioms 1 to 10 of Definition 6. The axioms other than Axioms 9 and 10 are satisfied by assumption.

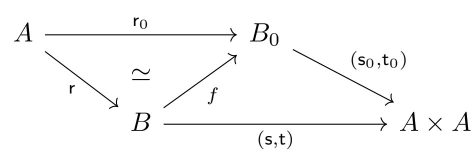
For Axiom 9, let $f : A \to B$ be an acyclic fibration in $\mathcal{E}$ and $v : QB \to QA$ a section of $Qf : QA \to QB$. One can factor f as a vertical acyclic fibration $f_1 : A \to A_1$ followed by a horizontal acyclic fibration $f_2 : A_1 \to B$. By the cartesianness of f_2, we have a section $g_2 : B \to A_1$ of f_2 over v. There exists a section $g_1 : A_1 \to A$ of $f_1 : A \to A_1$. Since f_1 is vertical, g_1 must be vertical. Then $g_1 g_2 : B \to A$ is a section of $f = f_2 f_1 : A \to B$ over v.

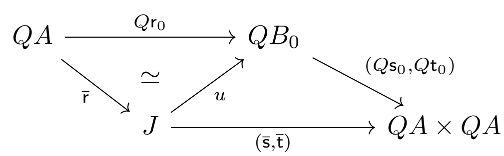
For Axiom 10, we first show the following *homotopy lifting property*.

Lemma 11. *Let $A \to C$ be an arrow and $B \to C$ a fibration in $\mathcal{E}$. For arrows $f : A \to B$ over C and $v : QA \to QB$ over QC such that $Qf \simeq_{QC} v$, there exists an arrow $g : A \to B$ over C and over v such that $f \simeq_C g$.*

Proof. Let $\mathrm{P}_C B$ be a path object of B over C. Since $Qf \simeq_{QC} v$, there exists an arrow $w : QA \to Q(\mathrm{P}_C B)$ such that $(Q\mathsf{s})w = Qf$ and $(Q\mathsf{t})w = v$. Since $\mathsf{s} : \mathrm{P}_C B \to B$ is an acyclic fibration, we have an arrow $h : A \to \mathrm{P}_C B$ over w such that $\mathsf{s}h = f$, using the section lifting property. Let $g = \mathsf{t}h : A \to B$. Then $f \simeq_C g$ by definition and we have $Qg = Q(\mathsf{t}h) = (Q\mathsf{t})w = v$. □

Let $A \in \mathcal{E}$ be an object and $(I, \bar{\mathsf{r}}, \bar{\mathsf{s}}, \bar{\mathsf{t}})$ a path object of QA. Let $(B_0, \mathsf{r}_0, \mathsf{s}_0, \mathsf{t}_0)$ be a path object of A. Since $\bar{\mathsf{r}} : QA \to I$ is a weak equivalence and $(Q\mathsf{s}_0, Q\mathsf{t}_0) : QB_0 \to QA \times QA$ is a fibration, we have an arrow $u : I \to QB_0$ over $QA \times QA$ such that $u\bar{\mathsf{r}} \simeq_{QA \times QA} Q\mathsf{r}_0$ by Theorem 3. Since $Q\mathsf{r}_0$ and $\bar{\mathsf{r}}$ are weak equivalences, so is u. Let $B = u^* B_0$ and $f : B \to B_0$ the cartesian arrow over u. Since u is a weak equivalence, so is f. By the stability of fibrations under reindexing, the composite $(\mathsf{s}, \mathsf{t}) := (\mathsf{s}_0, \mathsf{t}_0)f : B \to A \times A$ is a fibration over $(\bar{\mathsf{s}}, \bar{\mathsf{t}})$. By Lemma 11, there exists an arrow $\mathsf{r}_1 : A \to B_0$ over $A \times A$ and over $u\bar{\mathsf{r}}$ such that $\mathsf{r}_0 \simeq_{A \times A} \mathsf{r}_1$. Since r_0 is a weak equivalence, so is r_1. By the cartesianness of f, we have the unique arrow $\mathsf{r} : A \to B$ over $\bar{\mathsf{r}}$ such that $f\mathsf{r} = \mathsf{r}_1$. Since r_1 and f are weak equivalences, so is r. Then $(B, \mathsf{r}, \mathsf{s}, \mathsf{t})$ is a path object of A over $(I, \bar{\mathsf{r}}, \bar{\mathsf{s}}, \bar{\mathsf{t}})$.





□

2.2 Base Change

Proposition 12. *Let $\mathcal{E}$ be a fibred path category over a path category $\mathcal{B}$, $\mathcal{A}$ a path category and $F : \mathcal{A} \to \mathcal{B}$ a functor. If F preserves fibrations and weak equivalences, then the base change $F^*\mathcal{E}$ is a fibred path category over $\mathcal{A}$, where an arrow $(u, f) : (I, A) \to (J, B)$ in $F^*\mathcal{E} = \mathcal{A} \times_{\mathcal{B}} \mathcal{E}$ is a fibration (weak equivalence) if u is a fibration (weak equivalence) in $\mathcal{A}$ and f is a fibration (weak equivalence) in $\mathcal{E}$.*

Proof. Immediate from the definition. □

Proposition 13. *Let $\mathcal{E}$ be a fibred path category over a path category $\mathcal{B}$, $\mathcal{A}$ a path category, $F, G : \mathcal{A} \to \mathcal{B}$ functors that preserve fibrations and weak equivalences,* *and $\sigma : F \Rightarrow G$ a natural transformation. Then the fibred functor $\sigma^* : G^*\mathcal{E} \to F^*\mathcal{E}$ over $\mathcal{A}$ defined by the reindexing along σ is a fibred exact functor.*

Proof. Immediate from the definition. □

Corollary 14. *Let $\mathcal{E}$ be a fibred path category over a path category $\mathcal{B}$.*

1. *For any object $I \in \mathcal{B}$, the fibre $\mathcal{E}^I$ is a path category, where fibrations and weak equivalences are inherited from $\mathcal{E}$.*
2. *For any arrow $u : I \to J$ in $\mathcal{B}$, the reindexing functor $u^* : \mathcal{E}^J \to \mathcal{E}^I$ is an exact functor.*

Proof. Use Propositions 12 and 13 with functors from the terminal path category which automatically preserve fibrations and weak equivalences. □

2.3 Indexed Path Categories

Definition 15. Let $\mathcal{B}$ be a path category. A *$\mathcal{B}$-indexed path category* is a $\mathcal{B}$-indexed category $\mathbb{E}$ satisfying the following conditions:

- $\mathbb{E}^I$ is a path category for every object $I \in \mathcal{B}$;
- $u^* : \mathbb{E}^J \to \mathbb{E}^I$ is an exact functor for every arrow $u : I \to J$ in $\mathcal{B}$;
- $u^* : \mathbb{E}^J \to \mathbb{E}^I$ is a weak equivalence of path categories for every weak equivalence $u : I \to J$ in $\mathcal{B}$.

Definition 16. Let $\mathcal{B}$ be a path category and $\mathbb{E}$ and $\mathbb{F}$ be $\mathcal{B}$-indexed path categories. A *$\mathcal{B}$-indexed exact functor* $\mathbb{E} \to \mathbb{F}$ is a $\mathcal{B}$-indexed functor $S : \mathbb{E} \to \mathbb{F}$ such that $S^I : \mathbb{E}^I \to \mathbb{F}^I$ is an exact functor for every object $I \in \mathcal{B}$.

Proposition 17. *Let $\mathcal{B}$ be a path category and $\mathbb{E}$ a $\mathcal{B}$-indexed path category. Then the Grothendieck construction $\int_{\mathcal{B}} \mathbb{E}$, together with the projection $\int_{\mathcal{B}} \mathbb{E} \to \mathcal{B}$, is a fibred path category over $\mathcal{B}$ where an arrow $(u, f) : (I, A) \to (J, B)$ in $\int_{\mathcal{B}} \mathbb{E}$ is defined to be a fibration (weak equivalence) if $u : I \to J$ is a fibration (weak equivalence) in $\mathcal{B}$ and $f : A \to u^*B$ is a fibration (weak equivalence) in $\mathbb{E}^I$.*

Proof. Exercise. □

Lemma 18. *Let $\mathcal{E}$ be a fibred path category over a path category $\mathcal{B}$ and $u : I \to J$ a weak equivalence in $\mathcal{B}$. Then the reindexing functor $u^* : \mathcal{E}^J \to \mathcal{E}^I$ is a weak equivalence of path categories.*

Proof. Exercise. □

Theorem 19. *Let $\mathcal{B}$ be a path category. The Grothendieck construction $\mathbb{E} \mapsto \int_{\mathcal{B}} \mathbb{E}$ is part of a biequivalence between the 2-category of $\mathcal{B}$-indexed path categories and the 2-category of fibred path categories over $\mathcal{B}$.*

Proof. By Corollary 14 and Lemma 18, $(\int_{\mathcal{B}} -)$ is biessentially surjective on objects. To show that $(\int_{\mathcal{B}} -)$ is a local equivalence, it suffices to show that a $\mathcal{B}$-indexed functor $S : \mathbb{E} \to \mathbb{F}$ between $\mathcal{B}$-indexed path categories is a $\mathcal{B}$-indexed exact functor if and only if $\int_{\mathcal{B}} S : \int_{\mathcal{B}} \mathbb{E} \to \int_{\mathcal{B}} \mathbb{F}$ is a fibred exact functor over $\mathcal{B}$, which is left as an exercise. $\square$

References

[1] Benno van den Berg and Ieke Moerdijk. "Exact completion of path categories and algebraic set theory: Part I: Exact completion of path categories". In: *Journal of Pure and Applied Algebra* 222.10 (2018), pp. 3137–3181. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2017.11.017.