# Adding Classes to Set Theory

ZFC $\subseteq$ NBG $\subseteq$ MK

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# NBG

Von Neumann–Bernays–Gödel

# Differences between ZFC and NGB

The universe of ZFC consists of sets. (classes are formulas)

The universe of NBG consists of classes, some classes are sets.

**Theorem**

NBG is a conservative extension of ZFC, that is, they both prove the same statements about sets.

Note that this shows that ZFC and NBG are equiconsistent.

**Theorem**

NBG is finitely axiomatizable, ZFC is not.

# Background

ZFC is named after Ernst Zermelo and Abraham Fraenkel:

1908 Zermelo proposed the first axiomatic set theory,

1922 Fraenkel added the axiom of foundation and the axiom schema of replacement.

NBG is named after John von Neumann, Paul Bernays and Kurt Gödel:

1925 Neumann introduced classes into set theory, his theory used arguments and functions as primitive notions;

1929 Bernays reformulated to use sets and classes instead;

1931 Gödel simplified by making every set a class.

# Conventions

A set is a class that is an element of another class.

A proper class is a class that is not a set .

We use uppercase letters for classes and lowercase letters for sets.

To shorten notation we will write:

$$\exists x \phi(x) \text{ instead of } \exists X(\exists Y(X \in Y) \wedge \phi(X)),$$

$$\forall x \phi(x) \text{ instead of } \forall X(\exists Y(X \in Y) \to \phi(X)).$$

# Axioms (basics)

**Axiom (extensionality)**

$\forall A \forall B (\forall x (x \in A \leftrightarrow x \in B) \rightarrow A = B)$.

**Axiom (foundation)**

$\forall A (\exists (x \in A) \rightarrow \exists (x \in A) \forall (y \in A)(y \notin x))$.

# Axioms (sets)

### Axiom (pair)

$$\forall a \forall b \exists c \forall x (x \in c \leftrightarrow x = a \lor x = b).$$

### Axiom (union)

$$\forall a \exists b \forall x (x \in b \leftrightarrow \exists y (x \in y \land y \in a)).$$

### Axiom (power set)

$$\forall a \exists b \forall x (x \in b \leftrightarrow \forall y (y \in x \to y \in a)).$$

### Axiom (infinity)

$$\exists a (\exists (x \in a) \land \forall (x \in a) \exists (y \in a)(y \neq x \land \forall (z \in x)(z \in y)).$$

# Axioms (comprehension)

As usual we define $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$ and:

$$\langle x_1, \dots, x_n\rangle = \langle\langle x_1, \dots, x_{n-1}\rangle, x_n\rangle.$$

## Axiom Scheme (comprehension)

For every formula $\phi(\vec{A}, \vec{x})$ that only quantifies over sets:

$$\forall \vec{A} \exists B \forall \vec{x} (\langle \vec{x}\rangle \in B \leftrightarrow \phi(\vec{A}, \vec{x})).$$

# Axioms (replacement)

A function is a class $F$ such that we have:

$$\begin{aligned}\mathrm{fun}(F) = \forall(z \in F)(\exists x \exists y(z = \langle x, y \rangle)) \wedge \\ \forall x \forall y \forall z(\langle x, y \rangle \in F \wedge \langle x, z \rangle \in F \to y = z).\end{aligned}$$

**Axiom (replacement)**

$\forall F \forall a(\mathrm{fun}(F) \to \exists b \forall y(y \in b \leftrightarrow \exists(x \in a)(\langle x, y \rangle \in F))).$

**Axiom (global choice)**

$\exists F(\mathrm{fun}(F) \wedge \forall x(\exists(y \in x) \to \exists(y \in x)(\langle x, y \rangle \in F)).$

## Comparison of ZFC and NBG

| ZFC | NBG |
|---|---|
| extensionality | extensionality |
| foundation | foundation |
| pair | pair |
| union | union |
| power set | power set |
| infinity | infinity |
| scheme of seperation | scheme of comprehension |
| scheme of replacement | replacement |
| choice | global choice |

# Theorems

**Theorem (seperation)**

If $a$ is a set and $B \subseteq a$ then $B$ is a set.

*Proof.* With comprehension take $I_B = \{\langle x, x \rangle \mid x \in B\}$. Then the image of $a$ under $I_B$ is $B$ so by replacement $B$ is a set. $\square$

**Theorem (limitation of size)**

A class is a proper class iff it has a bijection to the universal class $V$.

*Proof Sketch.* By global choice every class $C$ has a well-ordering, this gives an injection $C \to \mathrm{Ord}$ whose image is an initial segment. If it is not surjective then $C$ is a set, otherwise $|C| = |\mathrm{Ord}|$. By the same reasoning we also get $|V| = |\mathrm{Ord}|$ so $|C| = |\mathrm{Ord}| = |V|$. $\square$

# NBG is finitely axiomatizable

**Theorem**

NBG is finitely axiomatizable.

The axiom scheme of comprehension gives infinitely many axioms, we show that we only need 7 of them.

To do this we show that we can prove every instance of comprehension using the other NBG axioms and these 7 instances.

# NBG is finitely axiomatizable: Axioms (formula)

**Axiom (complement/negation)**

$\forall A \exists B \forall x (x \in B \leftrightarrow x \notin A).$

**Axiom (union/disjunction)**

$\forall A \forall B \exists C \forall x (x \in C \leftrightarrow x \in A \vee x \in B).$

**Axiom (domain/existential)**

$\forall A \exists B \forall x (x \in B \leftrightarrow \exists y (\langle x, y \rangle \in A)).$

# NBG is finitely axiomatizable: Proof (preliminaries)

Suppose that we have classes $A_1, \dots, A_m$ and a formula $\phi(A_1, \dots, A_m, x_1, \dots, x_n)$ only quantifying over sets. We show:

$$\exists C_\phi \forall x_1 \dots \forall x_n (\langle x_1, \dots, x_n \rangle \in C_\phi \leftrightarrow \phi(A_1, \dots, A_m, x_1, \dots, x_n)).$$

First we transform $\phi$ to a more manageable equivalent form:

1. Replace $A_k \in \alpha$ with $\exists y(y = A_k \wedge y \in \alpha)$.
2. Replace $\alpha = \beta$ with $\forall z(z \in \alpha \leftrightarrow z \in \beta)$.
3. Replace $\wedge, \forall, \rightarrow, \leftrightarrow$ using $\neg, \vee, \exists$.
4. Replace $\exists y \psi(y)$ with $\exists x_{n+d} \psi(x_{x+d})$ where $d$ is the quantifier depth. Example: $\exists y(y \in x_1 \vee \exists z(z \in x_2)) \vee \exists z(z \in x_1)$ becomes $\exists x_3(x_3 \in x_1 \vee \exists x_4(x_4 \in x_2)) \vee \exists x_3(x_3 \in x_1)$.

# NBG is finitely axiomatizable: Proof (induction)

We construct $C_\phi$ using induction on the structure of $\phi$:

$$\begin{aligned}
\phi = x_i \in x_j &\implies C_\phi = E_{i,j,n} = \{\langle x_1, \dots, x_n\rangle \mid x_i \in x_j\}, \\
\phi = x_i \in A_k &\implies C_\phi = E'_{i,k,n} = \{\langle x_1, \dots, x_n\rangle \mid x_i \in A_k\}, \\
\phi = \neg\psi &\implies C_\phi = \complement C_\psi, \\
\phi = \psi \vee \chi &\implies C_\phi = C_\psi \cup C_\chi, \\
\phi = \exists x_{n+1}\psi &\implies C_\phi = \operatorname{dom}(C_\psi),
\end{aligned}$$

The only thing left to check is that we can construct $E_{i,j,n}$ and $E'_{i,k,n}$.

# NBG is finitely axiomatizable: Axioms (tuple)

**Axiom (membership)**

$\exists A \forall x \forall y (\langle x, y \rangle \in A \leftrightarrow x \in y)$.

**Axiom (product)**

$\forall A \forall B \exists C \forall z (z \in C \leftrightarrow \exists x \exists y (z = \langle x, y \rangle \wedge x \in A \wedge y \in B))$.

**Axiom (transpose)**

$\forall A \exists B \forall x \forall y \forall z (\langle x, z, y \rangle \in B \leftrightarrow \langle x, y, z \rangle \in A)$.

**Axiom (cycle)**

$\forall A \exists B \forall x \forall y \forall z (\langle y, z, x \rangle \in B \leftrightarrow \langle x, y, z \rangle \in A)$.

## NBG is finitely axiomatizable: Lemmas

### Lemma (tuple)

1 $\forall A \exists B \forall x \forall y (\langle y, x\rangle \in B \leftrightarrow \langle x, y\rangle \in A)$

2 $\forall A \exists B \forall x \forall y \forall z (\langle z, x, y\rangle \in B \leftrightarrow \langle x, y\rangle \in A)$

3 $\forall A \exists B \forall x \forall y \forall z (\langle x, z, y\rangle \in B \leftrightarrow \langle x, y\rangle \in A)$

4 $\forall A \exists B \forall x \forall y \forall z (\langle x, y, z\rangle \in B \leftrightarrow \langle x, y\rangle \in A)$

### Lemma (expansion)

If we have $P \subseteq V \times V$ and $i \neq j$ then we can construct

$$P_{i,j,n} = \{\langle x_1, \dots, x_n\rangle \mid \langle x_i, x_j\rangle \in P\}.$$

# NBG is finitely axiomatizable: Proof (E and E')

Note that by the membership axiom we have $E = \{\langle x, y\rangle \mid x \in y\}$.

We construct $E_{i,j,n} = \{\langle x_1, \dots, x_n\rangle \mid x_i \in x_j\}$ as follows:

- If $i = j$ take $E_{i,j,n} = \emptyset = \complement V$.
- If $i \neq j$ use the expansion lemma on $E$ to get $E_{i,j,n}$.

We construct $E'_{i,k,n} = \{\langle x_1, \dots, x_n\rangle \mid x_i \in A_k\}$ as follows:

- If $n = 1$ take $E'_{i,k,n} = Y_k$.
- If $n \neq 1$ use the expansion lemma on $Y_k \times V$ to get $E'_{i,k,n}$.

This completes the proof.

# MK

Morse–Kelley

# Differences between NBG and MK

There is only one difference between MK and NBG:

**Axiom Scheme (comprehension)**

For every formula $\phi(\vec{A}, \vec{x})$ ~~that only quantifies over sets~~:

$$\forall \vec{A} \exists B \forall \vec{x} (\langle \vec{x} \rangle \in B \leftrightarrow \phi(\vec{A}, \vec{x})).$$

**Theorem**

MK is not a conservative extension of ZFC.

**Theorem**

MK is not finitely axiomatizable.

## Background

MK is named after Anthony Morse and John Kelley:

- 1949 Wang first set out the theory.
- 1955 Kelley publicised a version of Morses theory in an appendix.
- 1965 Morse gave his version in an idiosyncratic formal language.

# Consistency

**Theorem**

MK can prove the consistency of ZFC and NBG.

**Theorem**

MK is equiconsistent with ZFC plus a strong inaccessiable cardianal.

# Discussion

How much stronger is MK than NBG?

How do we handle tuples for proper classes? We can still define pairs by taking $\langle C, D\rangle := (\{0\} \times C) \cup (\{1\} \times D)$.

How important is a finite axiomatization?

How do we want to handle collections which are too large?

- Ommit them like ZFC.
- Allow one more level like NBG and MK.
- Hierarchy of levels like type theory.