Conservativity of

Type Theory
over
Higher-order Arithmetic

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Overview

We delineate the arithmetic of dependent type theory:

- Classical Result: type theories without universes are conservative over Heyting Arithmetic (HA). (Beeson 1979)
- Our Result: type theories with a single level of universes are conservative over Higher-order Heyting Arithmetic (HAH).

The precise conservativity depends on our interpretation of logic:

- Proof-irrelevant: type theories prove the same arithmetical theorems as HAH (of any order).
- Proof-relevant: type theories prove different second-order but the same first-order arithmetical theorems as HAH.

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arithmetic $\rightleftarrows$ type theory

Higher-order Heyting Arithmetic

In higher-order logic we can quantify over powersets of the domain. If we write $\exists x^n$ or $\forall x^n$ then x is an element of the n-th powerset:

- x^0 is an element of the domain,
- x^1 is a set,
- x^2 is a set of sets,
- and so on.

For x^n and Y^{n+1} we have a new atomic formula: $x \in Y$.
We have two new axiom schemes:

$$\forall X^{n+1} \forall Y^{n+1} (\forall z^n (z \in X \leftrightarrow z \in Y) \to X = Y), \qquad \text{(extensionality)}$$

$$\exists X^{n+1} \, \forall z^n \, (z \in X \leftrightarrow \phi[z]). \qquad \text{(specification)}$$

HAH has the axioms of PA but in intuitionistic higher-order logic.

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Interpreting Logic in Type Theory

Using a universe $\mathcal{U}$, we can define powertypes:

$$\mathcal{P}\, A := A \to \mathcal{U}.$$

Proof-irrelevant interpretation, using propositional truncation:

$$\begin{aligned}
(A \vee B)^\bullet &:= \|A^\bullet + B^\bullet\|, & (\exists x^n\, B[x^n])^\bullet &:= \|\Sigma(x : \mathcal{P}^n\, \mathbb{N})\, B[x^n]^\bullet\|,\\
(A \wedge B)^\bullet &:= A^\bullet \times B^\bullet, & (\forall x^n\, B[x^n])^\bullet &:= \Pi(x : \mathcal{P}^n\, \mathbb{N})\, B[x^n]^\bullet,\\
(A \to B)^\bullet &:= A^\bullet \to B^\bullet. & &
\end{aligned}$$

Proof-relevant interpretation:

$$\begin{aligned}
(A \vee B)^\circ &:= A^\circ + B^\circ, & (\exists x^n\, B[x^n])^\circ &:= \Sigma(x : \mathcal{P}^n\, \mathbb{N})\, B[x^n]^\circ,\\
(A \wedge B)^\circ &:= A^\circ \times B^\circ, & (\forall x^n\, B[x^n])^\circ &:= \Pi(x : \mathcal{P}^n\, \mathbb{N})\, B[x^n]^\circ,\\
(A \to B)^\circ &:= A^\circ \to B^\circ. & &
\end{aligned}$$

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Type Theory

We show conservativity for a strong theory: a version $\lambda C+$ of the Calculus of Inductive Constructions (Coq/Lean), which extends Martin-Löf type theory (Agda).

Type constructors: $\mathbb{0}, \mathbb{1}, \mathbb{2}, \ldots, \mathbb{N}, \Sigma, \Pi, \mathrm{W}, =, \| \cdot \|, /$.
Impredicative universes Prop, Set : Type:

- So if A : Type and $x : A \vdash B$: Set then $\Pi(x : A)B[x]$: Set.
- Prop is proof-irrelevant: all terms of a P : Prop are equal.
- Set is proof-relevant: contains data types such as $\mathbb{N}$.

Extensionality, meaning that $=$ and $\equiv$ coincide, so:

- uniqueness of identity proofs,
- function extensionality.

We do not assume more universes Type_i or that Prop : Set.

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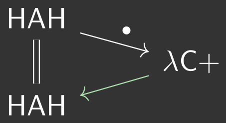
Main Result (proof-irrelevant)

Theorem (proof-irrelevant interpretation)

$\lambda C+$ proves the same formulas as HAH (of any order).

Proof Sketch. $\lambda C+$ derives the axioms and rules of HAH.
The difficult part is showing that it does not prove more.

We build a model for $\lambda C+$ using only concepts of HAH.
This gives us a realizability interpretation:



We show that the diagram commutes up to logical equivalence. □

Model

In our model we interpret:

$$\begin{aligned} \text{propositions} &\rightsquigarrow \text{subsingletons}, \\ \text{sets} &\rightsquigarrow \text{partial equivalence relations (PER's)}, \\ \text{types} &\rightsquigarrow \text{assemblies}. \end{aligned}$$

Variation on a well-known model for the Calculus of Constructions (Hyland1988, Reus1999), modified in two ways:

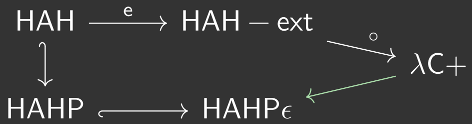
- we restrict sets to elements of some $\mathcal{P}^n\,\mathbb{N}$,
- we extend the interpretation to our larger theory λC+.

Main Result (proof-relevant)

Theorem (proof-relevant interpretation)

$\lambda C+$ proves distinct second, but the same first-order formulas as HAH.

Proof Sketch. $\lambda C+$ proves choice but not extensionality.
The following diagram commutes for first-order formulas:



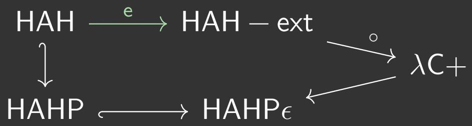
e interprets HAH in HAH − ext by inductively redefining $=$ and $\in$.
HAHP adds primitive notions for partial recursive functions.
HAHPϵ adds partial choice functions as Hilbert epsilon constants.
We show that HAHPϵ conservatively extends HAH. □

Main Result (proof-relevant)

Theorem (proof-relevant interpretation)

λC+ proves distinct second, but the same first-order formulas as HAH.

Proof Sketch.



e interprets HAH in HAH − ext by inductively redefining $=$ and $\in$:

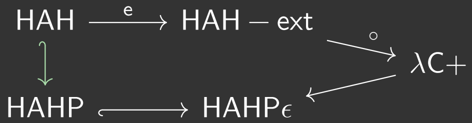
$$(x =^0_{\mathsf{e}} y) := (x =^0 y)$$
$$(X =^{n+1}_{\mathsf{e}} Y) := \forall z^n (z \in^n_{\mathsf{e}} X \leftrightarrow z \in^n_{\mathsf{e}} Y),$$
$$(x \in^n_{\mathsf{e}} Y) := \exists z^n (z =^n_{\mathsf{e}} x \wedge z \in^n Y).$$

Main Result (proof-relevant)

Theorem (proof-relevant interpretation)

λC+ proves distinct second, but the same first-order formulas as HAH.

Proof Sketch.



HAHP adds primitive notions for partial recursive functions:

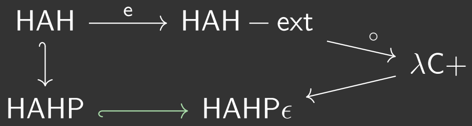
- We extend Beeson's logic of partial terms to higher-order logic.
- We add a primitive notion $\{x^0\}\, y^0$:
  intuitively the x-th partial recursive function applied to y.
- We add a primitive notion $t\downarrow$: the term t is defined.

Main Result (proof-relevant)

Theorem (proof-relevant interpretation)

λC+ proves distinct second, but the same first-order formulas as HAH.

Proof Sketch.



HAHPϵ adds partial choice functions as Hilbert epsilon constants:

- For every formula $\phi[\vec{x}, y]$ a new constant $\epsilon^0_{y.\phi}$ and axioms:

$$\forall \vec{x}\,(\exists y\,\phi[\vec{x}, y] \to \{\epsilon_{y.\phi}\}\,\vec{x}\downarrow),\ \forall \vec{x}\,(\{\epsilon_{y.\phi}\}\,\vec{x}\downarrow \to \phi[\vec{x}, \{\epsilon_{y.\phi}\}\,\vec{x}]).$$

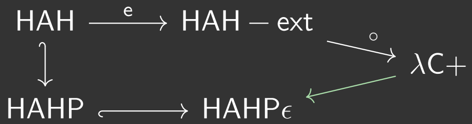
So, $\epsilon^0_{y.\phi}$ encodes a partial function sending $\vec{x}$ to a y with $\phi[\vec{x}, y]$.

Main Result (proof-relevant)

Theorem (proof-relevant interpretation)

$\lambda C+$ proves distinct second, but the same first-order formulas as HAH.

Proof Sketch. $\lambda C+$ proves choice but not extensionality.
The following diagram commutes for first-order formulas:



We show that HAHPϵ conservatively extends HAH:

- We use proof theoretic forcing, oracles, and compactness.

Generalisations and Summary

Our methods restrict to systems in the lambda cube to show

$$\lambda C+ \text{ is conservative over HAH,}$$
$$\lambda P2+ \text{ is conservative over HA2,}$$
$$\lambda P+ \text{ is conservative over HA,}$$

where the interpretation of logic determines the conservativity:

- proof-irrelevant: conservative for all higher-order formulas,
- proof-relevant: conservative for first but not second-order.