

Conservativity of  
Type Theory  
over  
Higher-order Arithmetic

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## Overview

We investigate the relation between **arithmetic** and **type theory**.

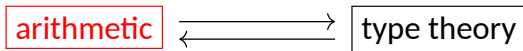
In dependent type theory, the number of **universes** influence how much can prove about the natural numbers:

- > **Classical Result**: MLO is conservative over HA (Beeson 1979).
- > **Our Result**: type theories with a single level of universes are conservative over Higher-order Heyting Arithmetic (HAH).

The amount of conservativity depends on our interpretation of logic:

- for **proof-irrelevant** versions: they prove exactly the same arithmetical theorems as HAH (of any order).
- for **proof-relevant** versions: they prove more second-order, but the same first-order arithmetical theorems as HAH.

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## Higher-order Heyting Arithmetic

In **higher-order logic** we can quantify over powersets of the domain.

If we write  $\exists x^n$  or  $\forall x^n$  then  $x$  is an element of the  $n$ -th powerset:

- >  $x^0$  is an element of the domain,
- >  $x^1$  is a set,
- >  $x^2$  is a set of sets,
- > and so on.

For  $x^n$  and  $Y^{n+1}$  we have a new **atomic formula**  $x \in Y$ .

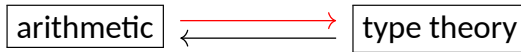
We have two additional logical **axiom schemes**:

$\forall X^{n+1}, Y^{n+1} (\forall z^n (z \in X \leftrightarrow z \in Y) \rightarrow X = Y)$ , (extensionality)

$\exists X^{n+1} \forall z^n (z \in X \leftrightarrow \phi[z])$ . (comprehension)

**HAH** has the axioms of **PA** but in **intuitionistic higher-order logic**.

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## Interpreting Logic in Type Theory

There are two interpretations of logic in type theory.

A **proof-irrelevant** way, using propositional truncation:

$$(A \vee B)^\bullet := \parallel A^\bullet + B^\bullet \parallel, \quad (\exists x^n B[x^n])^\bullet := \parallel \Sigma(x : \mathcal{P}^n \mathbb{N}) B[x^n]^\bullet \parallel,$$

$$(A \wedge B)^\bullet := A^\bullet \times B^\bullet, \quad (\forall x^n B[x^n])^\bullet := \Pi(x : \mathcal{P}^n \mathbb{N}) B[x^n]^\bullet,$$

$$(A \rightarrow B)^\bullet := A^\bullet \rightarrow B^\bullet.$$

And a **proof-relevant** way:

$$(A \vee B)^\circ := A^\circ + B^\circ, \quad (\exists x^n B[x^n])^\circ := \Sigma(x : \mathcal{P}^n \mathbb{N}) B[x^n]^\circ,$$

$$(A \wedge B)^\circ := A^\circ \times B^\circ, \quad (\forall x^n B[x^n])^\circ := \Pi(x : \mathcal{P}^n \mathbb{N}) B[x^n]^\circ,$$

$$(A \rightarrow B)^\circ := A^\circ \rightarrow B^\circ.$$

## Interpreting Powersets in Type Theory

Using a universe  $\mathcal{U}$ , we can define **powertypes**:

$$\mathcal{P} A := A \rightarrow \mathcal{U}.$$

We use this to interpret higher-order logic:

- > To satisfy **extensionality**, we need  $\mathcal{U}$  to satisfy:

$$\mathbf{funext} : \Pi(f, f' : \mathcal{P} A) (\Pi(x : A) (f x = f' x) \rightarrow (f = f')),$$

$$\mathbf{propext} : \Pi(P, P' : \mathcal{U}) ((P \leftrightarrow P') \rightarrow (P = P')).$$

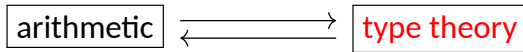
Alternatively, we can use setoids or quotients.

- > To satisfy **comprehension**, we need  $\mathcal{U}$  to be **impredicative**.

Alternatively, we can use propositional-resizing.

All alternatives give exactly the same conservativity results.

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## Type Theory

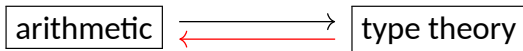
We work with a restriction  $\lambda C+$  of the Calculus of Inductive Constructions (the theory behind Coq and Lean).

We have two impredicative universes: **Prop**, **Set** : **Type**.

- > We use Prop to interpret propositions and powertypes.  
For this universe we assume **propext**.
- > We use Set for data types, and assume  $\mathbb{0}, \mathbb{1}, +, \Sigma, \Pi, =, \|A\|$ .  
We can also assume **W-types**, **uip**, or even **an extensional theory**.

Lastly, we assume **funext** in general.

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## Main Result (proof-irrelevant)

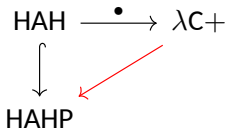
### Theorem (for a proof-irrelevant interpretation)

$\lambda C+$  proves the same arithmetical formulas as HAH (of any order).

*Proof Sketch.* We can show that  $\lambda C+$  proves the axioms of HAH. The difficult part is showing that it does not prove more.

We first give a conservative extension of HAH, named **HAHP**.

Then we construct an arrow:



$\overbrace{\lambda x b[x], \langle a, b \rangle}$

And show that the diagram commutes up to logical equivalence.  $\square$

## Realizability

We use realizability where we interpret:

propositions  $\rightsquigarrow$  subsingletons,

sets  $\rightsquigarrow$  partial equivalence relations (PERs),

types  $\rightsquigarrow$  assemblies or  $\omega$ -sets.

This takes inspiration from a well-known model for the calculus of constructions (Hyland1988, Reus1999), modified in two ways:

- > we restrict assemblies to those which live in some  $\mathcal{P}^n \mathbb{N}$ ,
- > we extend the interpretation to our extended theory.

## Main Result (proof-relevant)

### Theorem (for a proof-relevant interpretation)

$\lambda C+$  proves more second-order, but the same first-order arithmetical formulas as HAH.

*Proof Sketch.* An example of a second-order formula that is provable in type theory but not in HAH is the **axiom of choice**.

For first-order formulas we modify our earlier proof.

We extend HAHP to **HAHP $\epsilon$**  by adding a computable choice principle.

This theory is conservative over HAH, and we can construct an arrow:

$$\begin{array}{ccc} \text{HAH} & \xrightarrow{\circ} & \lambda C+ \\ \downarrow & \swarrow & \\ \text{HAHP}\epsilon & & \end{array}$$

s.t. the diagram commutes for first-order formulas up to equiv.  $\square$

## Generalisations and Summary

We can restrict our methods to systems in **the lambda cube** to show:

$\lambda C+$  is conservative over  $HAH$ ,

$\lambda P2+$  is conservative over  $HA2$ ,

$\lambda P+$  is conservative over  $HA$ ,

where the interpretation of logic determines the conservativity:

- for **proof-irrelevant** versions: conservativity for all formulas,
- for **proof-relevant** versions: conservativity for first-order, but not for second- or higher-order formulas.