Conservativity of

Type Theory

over

Higher-order Arithmetic

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Overview

We delineate the arithmetic of dependent type theory:

- Classical Result: type theories without universes are conservative over Heyting Arithmetic (HA). (Beeson 1979)
- Our Result: strong type theories with a single level of universes are conservative over Higher-order Heyting Arithmetic (HAH).

The amount of conservativity depends on our interpretation of logic:

- Proof-irrelevant: type theories prove the same arithmetical theorems as HAH (of any order).
- Proof-relevant: type theories prove different second-order but the same first-order arithmetical theorems as HAH.

Contents

arithmetic $\rightleftarrows$ type theory

Higher-order Heyting Arithmetic

In higher-order logic we can quantify over powersets of the domain.
If we write $\exists x^n$ or $\forall x^n$ then x is an element of the n-th powerset:

- x^0 is an element of the domain,
- x^1 is a set,
- x^2 is a set of sets,
- and so on.

For x^n and Y^{n+1} we have a new atomic formula: $x \in Y$.
We have two new axiom schemes:

$$\forall X^{n+1} \forall Y^{n+1} (\forall z^n (z \in X \leftrightarrow z \in Y) \to X = Y), \quad \text{(extensionality)}$$

$$\exists X^{n+1} \, \forall z^n \, (z \in X \leftrightarrow \phi[z]). \quad \text{(specification)}$$

HAH has the axioms of PA but in intuitionistic higher-order logic.

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Interpreting Logic in Type Theory

Using a universe $\mathcal{U}$, we can define powertypes:

$$\mathcal{P}\,A := A \to \mathcal{U}.$$

Proof-irrelevant interpretation, using propositional truncation:

$$(A \lor B)^\bullet := \|A^\bullet + B^\bullet\|, \quad (\exists x^n\, B[x^n])^\bullet := \|\Sigma(x : \mathcal{P}^n\, \mathbb{N})\, B[x^n]^\bullet\|,$$

$$(A \land B)^\bullet := A^\bullet \times B^\bullet, \quad (\forall x^n\, B[x^n])^\bullet := \Pi(x : \mathcal{P}^n\, \mathbb{N})\, B[x^n]^\bullet,$$

$$(A \to B)^\bullet := A^\bullet \to B^\bullet.$$

Proof-relevant interpretation:

$$(A \lor B)^\circ := A^\circ + B^\circ, \quad (\exists x^n\, B[x^n])^\circ := \Sigma(x : \mathcal{P}^n\, \mathbb{N})\, B[x^n]^\circ,$$

$$(A \land B)^\circ := A^\circ \times B^\circ, \quad (\forall x^n\, B[x^n])^\circ := \Pi(x : \mathcal{P}^n\, \mathbb{N})\, B[x^n]^\circ,$$

$$(A \to B)^\circ := A^\circ \to B^\circ.$$

Contents

arithmetic ⇄ type theory

Type Theory

We show conservativity for a strong theory:
a version λC+ of the Calculus of Inductive Constructions (`Coq/Lean`),
which is stronger than Martin-Löf type theory (`Agda`).

Type constructors: $\mathbb{0}, \mathbb{1}, \mathbb{2}, \ldots, \mathbb{N}, \Sigma, \Pi, \mathrm{W}, =, \|\cdot\|, /$.
Impredicative universes Prop, Set : Type:

- So if A : Type and $x : A \vdash B$: Set then $\Pi(x : A)B[x]$: Set.
- Prop is proof-irrelevant: all terms of a P : Prop are equal.
- Set is proof-relevant: contains data types such as $\mathbb{N}$.

Extensionality, meaning that $=$ and $\equiv$ coincide, so:

- uniqueness of identity proofs,
- function extensionality.

We do not assume more universes Type$_i$ or that Prop : Set.

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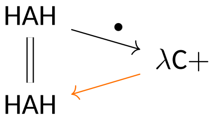
Main Result (proof-irrelevant)

Theorem (proof-irrelevant interpretation)

λC$+$ proves the same formulas as HAH (of any order).

Proof Sketch. λC$+$ derives the axioms and rules of HAH.
The difficult part is showing that it does not prove more.

We build a model for λC$+$ using only concepts of HAH.
This gives us a realizability interpretation:



We show that the diagram commutes up to logical equivalence. □

Model

In our model we interpret:

$$\begin{aligned} \text{propositions} &\rightsquigarrow \text{subsingletons}, \\ \text{sets} &\rightsquigarrow \text{partial equivalence relations (PER's)}, \\ \text{types} &\rightsquigarrow \text{assemblies}. \end{aligned}$$

Variation on a well-known model for the Calculus of Constructions (Hyland1988, Reus1999), modified in two ways:

- we restrict sets to elements of some $\mathcal{P}^n\,\mathbb{N}$,
- we extend the interpretation to our larger theory λC$+$.

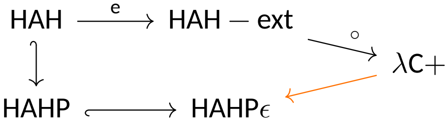
Main Result (proof-relevant)

Theorem (proof-relevant interpretation)

λC+ proves distinct second, but the same first-order formulas as HAH.

Proof Sketch. λC+ does not prove extensionality but it proves choice.

The following diagram commutes for first-order formulas:



e interprets HAH in HAH $-$ ext by inductively redefining $=$ and $\in$.

HAHP adds primitive notions for partial recursive functions.

HAHPϵ adds partial choice functions as Hilbert epsilon constants.

We show that HAHPϵ conservatively extends HAH. □

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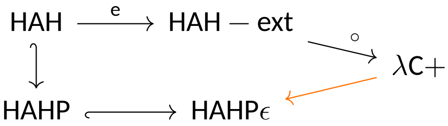

e interprets HAH in HAH $-$ ext by inductively redefining $=$ and $\in$:

$$(x =^0_{\mathsf{e}} y) := (x =^0 y)$$

$$(X =^{n+1}_{\mathsf{e}} Y) := \forall z^n (z \in^n_{\mathsf{e}} X \leftrightarrow z \in^n_{\mathsf{e}} Y),$$

$$(x \in^n_{\mathsf{e}} Y) := \exists z^n (z =^n_{\mathsf{e}} x \land z \in^n Y).$$

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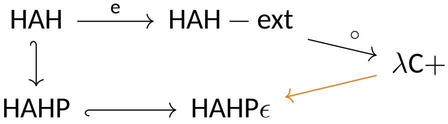

HAHP adds primitive notions for partial recursive functions:

- We extend Beeson's logic of partial terms to higher-order logic.
- We add a new primitive notion $\{x^0\}\, y^0$,
  intuitively the x-th partial recursive function applied to y.

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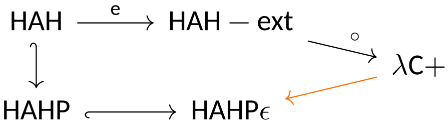

HAHPϵ adds partial choice functions as Hilbert epsilon constants:

- For every formula $\phi[\vec{x}, y]$ a new constant $\epsilon^0_{y.\phi}$ and axioms:

  $\forall \vec{x}\, (\exists y\, \phi[\vec{x}, y] \to \{\epsilon_{y.\phi}\}\, \vec{x}\downarrow),\ \forall \vec{x}\, (\{\epsilon_{y.\phi}\}\, \vec{x}\downarrow \to \phi[\vec{x}, \{\epsilon_{y.\phi}\}\, \vec{x}]).$

  So, $\epsilon^0_{y.\phi}$ encodes a partial function sending $\vec{x}$ to a y with $\phi[\vec{x}, y]$.

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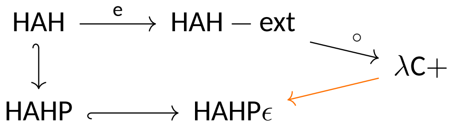

We show that HAHPϵ conservatively extends HAH:

- We use proof theoretic forcing, oracles, and compactness.

Generalisations and Summary

Our methods restrict to systems in the lambda cube to show

λC$+$ is conservative over HAH,

λP2$+$ is conservative over HA2,

λP$+$ is conservative over HA,

where the interpretation of logic determines the conservativity:

- proof-irrelevant: conservative for all higher-order formulas,
- ○ proof-relevant: conservative for first but not second-order.

Thank you!