De Jongh's Theorem for Type Theory

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Overview

We investigate the relation between arithmetic and type theory.

We compare:

- Second-order Heyting Arithmetic (HA2),
- Second-order Propositional Lambda Calculus (λP2), along with some additional assumptions ($\underbrace{\lambda\text{P}2+}_{\text{ind, uip, funext}}$).

Our main results are that λP$2+$ proves:

- the same first-order arithmetical formulas as HA2,
- more second-order arithmetical formulas than HA2.

This allows us to translate De Jongh's Theorem from HA2 to λP$2+$.

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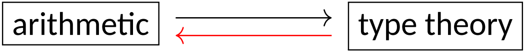


Arithmetic

In second-order logic we can quantify over n-ary relation symbols:

$$\exists X^n, \qquad\qquad \forall X^n.$$

HA2 is the constructive second-order theory with:

a constant 0, a unary function symbol S,

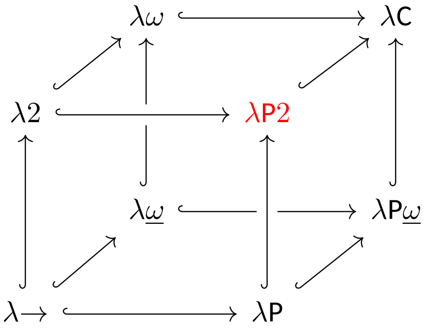
and axioms

$$\forall x\, (\mathsf{S}(x) \neq 0),$$
$$\forall x\, \forall y\, (\mathsf{S}(x) = \mathsf{S}(y) \to x = y),$$
$$\forall X^1\, (X(0) \wedge \forall x\, (X(x) \to X(\mathsf{S}(x))) \to \forall x\, X(x)).$$

Addition and multiplication can be defined in HA2.

The Lambda Cube

λP2 is part of a larger family of type theories (the lambda cube):



Our choice for λP2 has two justifications:

- it is the minimal theory that can interpret arithmetical formulas,
- it is the maximal theory that can be realized in arithmetic.

Basic Types

In λP2 we can define the following terms and types:

$$
\begin{aligned}
& \mathbb{0}, && \text{(empty type)} \\
* \ &: \ \mathbb{1}, && \text{(unit type)} \\
0,\ \mathsf{S}\,n \ &: \ \mathbb{N}, && \text{(natural numbers)} \\
\mathsf{in}_0\,a,\ \mathsf{in}_1\,b \ &: \ A + B, && \text{(disjoint union)} \\
\langle a, b\rangle \ &: \ A \times B, && \text{(Cartesian product)} \\
\lambda(x : A)\,b(x) \ &: \ A \to B, && \text{(function space)} \\
\langle a, b\rangle \ &: \ \Sigma(x : A)\,B(x), && \text{(dependent Cartesian product)} \\
\lambda(x : A)\,b(x) \ &: \ \Pi(x : A)\,B(x). && \text{(dependent function space)}
\end{aligned}
$$

Size

λP2 makes a distinction between small and large types.

The types we have seen until now are all small.

In λP2 there exists a large type Type_0 that contains all small types.

The following types also exist if either A or B is large:

$$A \to B, \qquad \text{(function space)}$$

$$\Pi(x : A)\, B(x). \qquad \text{(dependent function space)}$$

These types have the same size as B.

Formulas as Types

We can interpret arithmetical formulas as types (Curry-Howard):

$$\begin{aligned}
\bot &\text{ as } \mathbb{0}, \\
\top &\text{ as } \mathbb{1}, \\
A \vee B &\text{ as } A + B, \\
A \wedge B &\text{ as } A \times B, \\
A \to B &\text{ as } A \to B, \\
\exists x\, B(x) &\text{ as } \Sigma(x : \mathbb{N})\, B(x), \\
\forall x\, B(x) &\text{ as } \Pi(x : \mathbb{N})\, B(x), \\
\exists X^n\, B(X) &\text{ as } \Sigma(X : \mathbb{N} \times \cdots \times \mathbb{N} \to \mathsf{Type}_0)\, B(X), \\
\forall X^n\, B(X) &\text{ as } \Pi(X : \mathbb{N} \times \cdots \times \mathbb{N} \to \mathsf{Type}_0)\, B(X).
\end{aligned}$$

We say that λP2 proves the formula if the type is non-empty.

Main Results

Theorem (first-order)

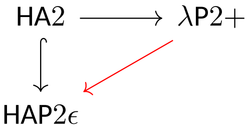
λP$2+$ proves the same first-order arithmetical formulas as HA2.

Proof Sketch. We can show that λP$2+$ proves the axioms of HA2.
The difficulty is showing that λP$2+$ does not prove more than HA2.

We first give a conservative extension of HA2, named HAP2ϵ.

$\overbrace{\lambda x\, b(x), \langle a,b \rangle}$ comp choice

Then we construct an arrow:



And we show that the diagram commutes for first-order formulas (up to logical equivalence). □

Main Results

Theorem (second-order)

λP$2+$ proves more second-order arithmetical formulas than HA2.

Proof Sketch. We consider the axiom of choice:

$$\forall Z^2\,(\forall x\,\exists y\,Z(x,y) \to \\ \exists F^2\,(\forall x\,\exists!y\,F(x,y) \land \forall x\,\forall y\,(F(x,y) \to Z(x,y))).$$

This is provable in λP$2+$ but not in HA2. □

De Jongh's Theorem

Now we can translate De Jongh's Theorem from HA2 to λP$2+$.

De Jongh's Theorem for HA2

Suppose that a propositional formula $A(P_0, \ldots, P_{n-1})$ is not provable in constructive propositional logic. Then there exist first-order arithmetical sentences $B_0, \ldots, B_{n-1}$ such that:

$$\text{HA}2 \nvdash A(B_0, \ldots, B_{n-1}).$$

Example: Consider the law of the excluded middle $P \vee \neg P$.
There exists a first-order arithmetical sentence B such that:

$$\text{HA}2 \nvdash B \vee \neg B.$$

Because λP$2+$ and HA prove the same first-order arithmetical formulas, we see that the theorem also holds for λP$2+$ (and λP2).

Conclusion

We have seen that λP$2+$ proves:

- the same first-order arithmetical formulas as HA2,
- more second-order arithmetical formulas than HA2.

This shows that De Jongh's Theorem also holds for λP2$+$:
the propositional logic of λP2$+$ is constructive.

This also shows that the propositional logic of every smaller type theory is constructive: simply-typed lambda calculus, system F, λP2.

Future work:

- Does λP2 already proof more arithmetical formulas than HA2?
- Is the first-order logic of HA2 constructive? (then also of λP$2+$)