Fixpoint Logic

μ-calculus

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Introduction

Fact

The bisimulation invariant fragment of FOL is modal logic.
The bisimulation invariant fragment of MSO is modal μ-calculus.

Theorem

Satisfiability for modal μ-calculus is decidable.

Modal mu-calculus

Syntax

$$\phi \; ::= \; \top \mid \bot \mid p \mid \overline{p} \mid \phi \vee \psi \mid \phi \wedge \psi \mid \Diamond\phi \mid \Box\phi \mid \mu x.\phi(x) \mid \nu x.\phi(x)$$

Note: x can only appear positively in $\phi(x)$.

Semantics

For a Kripke model $M = (S, R, V)$ we define:

$M, s \Vdash \mu x.\phi(x)$ iff $s \in T$ for the minimal $T \subseteq S$ with $T = [\![\phi(P_T)]\!]$;
$M, s \Vdash \nu x.\phi(x)$ iff $s \in T$ for the maximal $T \subseteq S$ with $T = [\![\phi(P_T)]\!]$.

where P_T is a new proposition that holds precisely on T.

Example

Semantics

$M, s \Vdash \mu x.\phi(x)$ iff $s \in T$ for the minimal $T \subseteq S$ with $T = [\![\phi(P_T)]\!]$.

What is $\mu x.p \vee \Diamond x$?

Idea: build up from $\emptyset$.

$$\begin{aligned} T_0 &:= [\![\bot]\!], \\ T_1 &:= [\![p \vee \Diamond \bot]\!], \\ T_2 &:= [\![p \vee \Diamond(p \vee \Diamond \bot)]\!], \\ &\vdots \\ T_\omega &:= \bigcup_{n \geq 0} T_n = [\![\Diamond^* p]\!] \end{aligned}$$

Note: not expressible in modal logic!

Example

Semantics

$M, s \Vdash \nu x.\phi(x)$ iff $s \in T$ for the maximal $T \subseteq S$ with $T = [\![\phi(P_T)]\!]$.

What is $\nu x.p \vee \Diamond x$?

Idea: break down from S.

$$\begin{aligned} T_0 &:= [\![\top]\!], \\ T_1 &:= [\![p \vee \Diamond\top]\!], \\ T_2 &:= [\![p \vee \Diamond(p \vee \Diamond\top)]\!], \\ &\vdots \\ T_\omega &:= \bigcap_{n \geq 0} T_n = [\![\Diamond^* p \vee \Diamond^\omega \top]\!] \end{aligned}$$

Note: not expressible in modal logic!

Game semantics

$M, s \Vdash \phi$ iff (s, ϕ) is a winning position for $\top$ in the following game:

Position	**Next move (if any)**
$(s, \top)$	$\top$ wins,
$(s, \bot)$	$\bot$ wins,
(s, p)	$\top$ wins iff $s \in V(p)$,
$(s, \overline{p})$	$\bot$ wins iff $s \in V(p)$,
$(s, \phi \vee \psi)$	$\top$ can play (s, ϕ) or (s, ψ),
$(s, \phi \wedge \psi)$	$\bot$ can play (s, ϕ) or (s, ψ),
$(s, \Diamond\phi)$	$\top$ can play (t, ϕ) with sRt,
$(s, \Box\phi)$	$\bot$ can play (t, ϕ) with sRt,
$(s, \mu x.\phi_x(x))$	automatic move to $(s, \phi_x(x))$,
$(s, \nu x.\phi_x(x))$	automatic move to $(s, \phi_x(x))$,
(s, x)	automatic move to $(s, \phi_x(x))$.

Example

How does this work for $\mu x.p \vee \Diamond x$?

Position	**Next move (if any)**
$(s, \top)$	$\top$ wins,
$(s, \bot)$	$\bot$ wins,
(s, p)	$\top$ wins iff $s \in V(p)$,
$(s, \overline{p})$	$\bot$ wins iff $s \in V(p)$,
$(s, \phi \vee \psi)$	$\top$ can play (s, ϕ) or (s, ψ),
$(s, \phi \wedge \psi)$	$\bot$ can play (s, ϕ) or (s, ψ),
$(s, \Diamond \phi)$	$\top$ can play (t, ϕ) with sRt,
$(s, \Box \phi)$	$\bot$ can play (t, ϕ) with sRt,
$(s, \mu x.\phi_x(x))$	automatic move to $(s, \phi_x(x))$,
$(s, \nu x.\phi_x(x))$	automatic move to $(s, \phi_x(x))$,
(s, x)	automatic move to $(s, \phi_x(x))$.

Infinite games

For an infinite game:

- if the (outermost) looped variable is a μ-variable then $\bot$ wins,
- if the (outermost) looped variable is a ν-variable then $\top$ wins.

Bisimulation invariance

Theorem

Modal μ-calculus is bisimulation invariant.

Proof Sketch. Suppose $M, s \,\underline{\leftrightarrow}\, M', s'$ and $M, s \Vdash \phi$. Then (s, ϕ) is a winning position for $\top$.

Key idea: Use bisimulation to mimic $\top$'s winning strategy for (s, ϕ) to obtain a winning strategy for (s', ϕ).

Decidability

Theorem (Grädel)

Model μ-calculus has the Lowenheim-Skolem property.

Corollary

If a formula is satisfiable then it is satisfiable in a countable tree model.

Theorem (Rabin)

The MSO theory of all countable trees is decidable.

Corollary

Satisfiability in the μ-calculus is decidable.

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