Models for Propositional Type Theory

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Equality

Intensional type theory (ITT) has two notions of equality:

definitional ($\equiv$)	external	reductions	decidable
propositional ($=$)	internal	proofs	undecidable

So, definitional eq forms a decidable fragment of propositional eq.

Why this particular decidable fragment?

There are reasons to consider larger or smaller fragments.

Larger Fragments

Consider addition of natural numbers. If we define:

$$0 + n \equiv n,$$
$$\mathsf{S}\, m + n \equiv \mathsf{S}\,(m + n).$$

Then we can prove:

$$m + 0 = m,$$
$$m + \mathsf{S}\, n = \mathsf{S}\,(m + n).$$

But we do not have definitional equalities.

We could add these without losing decidability.

Agda allows us to add proven equalities as new reductions.

This helps us avoid ‘transport hell’.

Does this allow us to prove more?

Smaller Fragments

Sometimes we have less reductions:

- We can define $\mathbb{N}$ as a W-type, however it will only satisfy the propositional β-rules.
- We can define M-types using $\mathbb{N}$ and function extensionality, however it will only satisfy the propositional β-rule. In fact, adding the definitional β-rule for M-types makes type checking undecidable.
- Cubical type theory only has a propositional β-rule for paths.

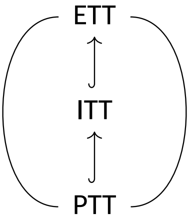
Can we still prove the same?

Extremes

To answer these questions, we consider the two extremes:

- Propositional type theory (PTT), without any definitional eq.
- Extensional type theory (ETT), where every eq is definitional.

These form the min and max of a lattice with ITT in the middle:



Conservativity

So, does ETT prove more than PTT? Yes, namely:

- function extensionality (funext),
- uniqueness of identity proofs (uip).

However, these are the only additional things we can prove:

- Hofmann (95) first gave a categorical proof that:
  ETT is conservative over ITT + funext + uip.
- Oary (05) gave a more syntactical proof.
- Winterhalter (19) turned the proof into an effective translation:
  ETT is conservative over PTT + funext + uip + 12 rules.
- Spadetto (23) showed in a setting without universes and W-types:
  ETT is conservative over PTT + funext + uip.

Generalisations

This still leaves open questions:

- Can we generalise this to other nodes in the lattice:
  when is it conservative to add more definitional eq?
- Univalence contradicts uip. So, how much definitional eq can we add without proving uip?

Bocquet (20) answers some of these questions:

- we can take a model for type theory with less definitional eq,
- take a quotient, and
- turn it into a model for type theory with more definitional eq.

Trade-offs

There are some trade-offs for the amount of definitional eq:

- more makes it easier to work inside the system,
- less makes is easier to find models for the system.

We will focus on models. What are models for PTT?

We compare two categorical notions:

- comprehension categories: more traditional and well-studied;
- path categories: introduced by Van den Berg and Moerdijk (17), more concise, taking inspiration from abstract homotopy theory.

We will show an equivalence between these two notions.

Categorical Semantics

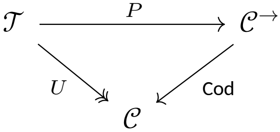
There are many different categorical semantics for type theory:

non-split	split
comprehension categories	split comprehension categories
display map categories	categories with attributes
	categories with families

The notions in each column are equivalent.

Our goal is to show that path categories fall in the first column.

Comprehension Categories



A comprehension category consists of:

- a category $\mathcal{C}$ of contexts,
- a category $\mathcal{T}$ of types,
- a fibration $U : \mathcal{T} \to \mathcal{C}$ sending every type to its context,
- a full and faithful functor $P : \mathcal{T} \to \mathcal{C}^{\to}$ sending every type A in context Γ to the display map $P_A : \Gamma.A \to \Gamma$.

A term of type A in context Γ is an $a : \Gamma \to \Gamma.A$ s.t. $P_A \circ a = \mathrm{id}_\Gamma$.

Substitution

That $U : \mathcal{T} \to \mathcal{C}$ is a fibration means that we can do substitution:

- for a type A in context Γ and a context morphism $\sigma : \Delta \to \Gamma$, there exists a type $A[\sigma]$ in context Δ and a pullback square:

$$\begin{array}{ccc} \Delta.A[\sigma] & \xrightarrow{\sigma.A} & \Gamma.A \\ {\scriptstyle P_{A[\sigma]}}\downarrow & \lrcorner & \downarrow{\scriptstyle P_A} \\ \Delta & \xrightarrow[\sigma]{} & \Gamma \end{array}$$

However, in general, we cannot pick every $A[\sigma]$ such that:

$$A[\mathrm{id}] = A,$$

$$A[\tau \circ \sigma] = A[\sigma][\tau].$$

A comprehension category with compatible choices is called split.

Splitting

A comprehension category has to be split to model type theory.

Luckily, there are ways to split comprehension categories:

$$\mathrm{SplitCompCat} \underset{}{\overset{}{\uparrow} \dashv \downarrow \dashv \uparrow} \mathrm{CompCat}$$

We are mostly interested in the left adjoint:

- Lumsdaine and Warren (14): Local Universe Construction.
- Bocquet (21): Generic Contexts.

Using our equivalence we can use this to split path categories.

Identity Types: Formation and Introduction

Comprehension categories only model the basic structure of dependent type theory. Each type former gives more requirements.

In this talk we focus on one type former: identity types.

The requirements are translated from the inference rules.
First, we have the formation and introduction rules:

$$\frac{\Gamma \vdash A : \mathsf{Type}}{\begin{array}{c}\Gamma, x : A, y : A \vdash x =_A y : \mathsf{Type} \\ \Gamma, x : A \vdash \mathsf{refl}_x : x =_A x\end{array}}$$

So, we require for every type A in context Γ:

- a type Id_A in context $\Gamma.A.A$,
- a term $\mathsf{refl}_A : \Gamma.A \to \Gamma.A.\mathsf{Id}_A[v_A]$.

Identity Types: Elimination and β-reduction

In ITT we have the elimination and β-reduction rules:

$$\frac{\begin{array}{l}\Gamma, x : A, y : A, p : x = y \vdash C[x, y, p] : \mathsf{Type} \\ \Gamma, x : A \vdash c[x] : C[x, x, \mathsf{refl}_x]\end{array}}{\begin{array}{l}\Gamma, x : A, y : A, p : x = y \vdash j_{c,p} : C[x, y, p] \\ \Gamma, x : A \vdash j_{c,\mathsf{refl}_x} \equiv c[x] : A\end{array}}$$

However, without Π-types we have to strengthen this to:

$$\frac{\begin{array}{l}\Gamma, x : A, y : A, p : x = y, \Delta[x, y, p] \vdash C[x, y, p] : \mathsf{Type} \\ \Gamma, x : A, \Delta[x, x, \mathsf{refl}_x] \vdash c[x] : C[x, x, \mathsf{refl}_x]\end{array}}{\begin{array}{l}\Gamma, x : A, y : A, p : x = y, \Delta[x, y, p] \vdash j_{c,p} : C[x, y, p] \\ \Gamma, x : A, \Delta[x, x, \mathsf{refl}_x] \vdash j_{c,\mathsf{refl}_x} \equiv c[x] : A\end{array}}$$

Identity Types: Elimination and β-reduction

In ITT we have the elimination and β-reduction rules:

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For PTT we have to change the β-reduction rule:

$$\frac{\Gamma, x : A, y : A, p : x = y, \Delta[x, y, p] \vdash C[x, y, p] : \mathsf{Type} \qquad \Gamma, x : A, \Delta[x, x, \mathsf{refl}_x] \vdash c[x] : C[x, x, \mathsf{refl}_x]}{\Gamma, x : A, y : A, p : x = y, \Delta[x, y, p] \vdash j_{c,p} : C[x, y, p] \qquad \Gamma, x : A, \Delta[x, x, \mathsf{refl}_x] \vdash h_{c,x} : j_{c,\mathsf{refl}_x} = c[x]}$$

Identity Types: Elimination and β-reduction

In PTT we have the elimination and β-reduction rules:

$$\frac{\begin{array}{l}\Gamma, x : A, y : A, p : x = y, \Delta[x, y, p] \vdash C[x, y, p] : \mathsf{Type} \\ \Gamma, x : A, \Delta[x, x, \mathsf{refl}_x] \vdash c[x] : C[x, x, \mathsf{refl}_x]\end{array}}{\begin{array}{l}\Gamma, x : A, y : A, p : x = y, \Delta[x, y, p] \vdash j_{c,p} : C[x, y, p] \\ \Gamma, x : A, \Delta[x, x, \mathsf{refl}_x] \vdash h_{c,x} : j_{c,\mathsf{refl}_x} = c[x]\end{array}}$$

The requirements on comprehension categories get complicated.

Checking that a category has this structure can be difficult.

Stability

In addition, we need choices that are stable under substitution:

$$\mathsf{Id}_A[\sigma] = \mathsf{Id}_{A[\sigma]},$$
$$\mathsf{refl}_A[\sigma] = \mathsf{refl}_{A[\sigma]},$$
$$j_{A,C,c}[\sigma] = j_{A[\sigma],C[\sigma],c[\sigma]},$$
$$h_{A,C,c}[\sigma] = h_{A[\sigma],C[\sigma],c[\sigma]}.$$

Fortunately, when we split the comprehension category, we also turn weakly stable structure into stable structure.

We have weakly stable propositional identity types if for every type A there exist an identity type $(\mathsf{Id}_A, \mathsf{refl}_A, j_A, h_A)$ s.t. for every σ:

- there exist j and h s.t. $(\mathsf{Id}_A[\sigma], \mathsf{refl}_A[\sigma], j, h)$ is an identity type.

Constructing Models

Suppose that we have a type theory TT with some definitional eq.

Then we can obtain a model for TT as follows:

- start with a path category;
- turn it into a comprehension category with weakly stable propositional identity types;
- split it to turn it into a split comprehension category with stable propositional identity types (a model of PTT);
- take a quotient to turn it into a model for TT.

This builds on the work of Winterhalter and Bocquet.

Path Categories

A path category is a category $\mathcal{C}$ with two classes of maps:

- fibrations: closed under pullbacks and compositions,
- weak equivalences, satisfying 2-out-of-6: if we have

  $$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

  where $g \circ f$ and $h \circ g$ are weak equivalences,
  then f, g, h, and $h \circ g \circ h$ are weak equivalences.

If a map is both then we call it an acyclic fibration:

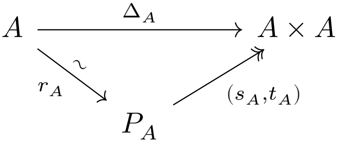
- every isomorphism is an acyclic fibration,
- pullbacks of acyclic fibrations are acyclic fibrations,
- every acyclic fibration has a section.

$\mathcal{C}$ has a terminal object 1 and every map $A \to 1$ is a fibration.

Path Objects

Lastly, a path category has a path object for every object A:

- a factorisation of the diagonal $\Delta_A = (\mathrm{id}_A, \mathrm{id}_A)$:



  into a weak equivalence r_A followed by a fibration (s_A, t_A).

We can use path objects to show that every morphism factors as a weak equivalence followed by a fibration.

Homotopy Equivalence

We call two maps $f, g : A \to B$ homotopic, written $f \simeq g$, if there exists a path object P_B for B and a map $h : A \to P_B$ such that we have $s_B \circ h = f$ and $t_B \circ h = g$.

We call $f : A \to B$ an homotopy equivalence, if there exists a map $g : B \to A$ such that $g \circ f \simeq \mathrm{id}_A$ and $f \circ g \simeq \mathrm{id}_B$.

The homotopy equivalences are precisely the weak equivalences.

From a Path Category to a Comprehension Category

Suppose that we have a path category $\mathcal{C}$.

If we take the full subcategory $\mathcal{C}^{\mathsf{fib}} \subseteq \mathcal{C}^{\rightarrow}$ of fibrations as the category of types, then we get a comprehension category:

$$\begin{array}{ccc} \mathcal{C}^{\mathsf{fib}} & \hookrightarrow & \mathcal{C}^{\rightarrow} \\ & {\scriptstyle \mathsf{Cod}} \searrow \qquad \swarrow {\scriptstyle \mathsf{Cod}} & \\ & \mathcal{C} & \end{array}$$

We will show that it has additional structure:

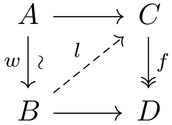
- weakly stable propositional identity types,
- weakly stable strong intensional Σ-types,
- democracy.

Weakly Stable Propositional Identity Types

For a type A we take $\mathsf{Id}_A := P_A$ and $\mathsf{refl}_A := r_A : A \to P_A$.

The elimination and β-reduction rules follow from a lifting theorem. For the simplified versions, without Δ, we use the theorem:

- if we have a commutative square



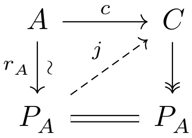
  where w is a weak equivalence and f is a fibration, then there exists an $l : B \to C$ such that the lower triangle commutes and the upper triangle commutes up to homotopy.

Weakly Stable Propositional Identity Types

For a type A we take $\mathrm{Id}_A := P_A$ and $\mathrm{refl}_A := r_A : A \to P_A$.

The elimination and β-reduction rules follow from a lifting theorem. For the simplified versions, without Δ, we use the theorem:

- For a type C over P_A and a term $c : A \to C$ we have



  so there exists a $j : C \to B$ such that the lower triangle commutes and the upper triangle commutes up to homotopy.

We get weak stability because we can prove that path objects are closed under pullbacks.

Weakly Stable Strong Intensional Σ-Types

A type theory has strong intensional Σ-types if it has Σ-types where both the β-reduction and η-reduction rules are definitional.

The requirements on a comprehension category can be simplified to:

- for every type A in context Γ and type B in context $\Gamma.A$, we have a type Σ_A^B in context Γ and an isomorphism $\Gamma.A.B \cong \Gamma.\Sigma_A^B$ such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma.A.B & \overset{\sim}{=\!=\!=} & \Gamma.\Sigma_A^B \\ \downarrow & & \downarrow \\ \Gamma.A & \longrightarrow & \Gamma \end{array}$$

Our path category satisfies this because the composition of two fibrations is a fibration.

Democracy

A comprehension category is called democratic if the category of contexts has a terminal object 1 and for every context Γ there exist:

- a type A_0 in context 1,
- a type A_1 in context $1.A_0$,
- a type A_2 in context $1.A_0.A_1$,

  $\vdots$
- a type A_{n-1} in context $1.A_0.\ldots.A_{n-2}$,

such that $\Gamma \cong 1.A_0.\ldots.A_{n-1}$.

Our path category satisfies this because it has a terminal object 1 and every map $\Gamma \to 1$ is a fibration.

From a Comprehension Category to a Path Category

Suppose that we have a democratic comprehension category $\mathcal{C}$ with weakly stable propositional identity types.

Then we can turn $\mathcal{C}$ into a path category by taking:

- the fibrations as the smallest class containing the display maps and closed under isomorphism and composition,
- the weak equivalences as the homotopy equivalences.

Then $\mathcal{C}$ has path objects because it has propositional identity types.

If we restrict ourselves to comprehension categories with weakly stable strong intensional Σ-types then this is a quasi-inverse of the functor sending path categories to comprehension categories.

Equivalence

Main Theorem

Path categories are equivalent to comprehension categories with:

- weakly stable propositional identity types,
- weakly stable strong intensional Σ-types,
- democracy.

This answers the question

- how can we turn path categories into models of PTT?

because we can split comprehension categories.

Open Questions

We end this talk with some open questions:

- What are the right axioms for PTT with universes?
  - Without universes it seems straightforward: we just make the β-reduction rule propositional.
  - With universes, we need Winterhalter's 12 rules to show that ITT is conservative over PTT.
  - Can these be simplified or already proven in PTT?
- Can we weaken path categories so that the corresponding comprehension category only has propositional Σ-types?
  - We have tried the following change: composisitions of fibrations are only equivalent to another fibration.
  - A lot of the nice properties of path categories seem to be lost.
- Can we simplify other type formers as we did with identity-types?

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