Unravelling Abstract Cyclic Proofs into Proofs by Induction

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Cyclic proofs

Cyclic proof systems replace (co)induction rules with circular reasoning.

Example. Consider the language of arithmetic with axioms:

$$0 + x = x, \qquad \mathsf{suc}\, x + y = \mathsf{suc}(x + y).$$

We have a cyclic proof:

$$\dfrac{\dfrac{}{\mathsf{suc}\, x + 0 = \mathsf{suc}\,(x + 0)}\,\text{axiom} \qquad x + 0 = x}{\vdots}$$
$$\dfrac{\dfrac{}{0 + 0 = 0}\,\text{axiom} \qquad \mathsf{suc}\, x + 0 = \mathsf{suc}\, x}{x + 0 = x}\,\text{case}_x$$

Upshot: instead of guessing an induction hypothesis, we generate a proof until we find a repeat with progress.

Soundness of cyclic proofs

For a cyclic proof system, we need to specify which cycles are allowed:

- we want to be restrictive enough to be sound;
- we want to be admissive enough to be complete, and easy to use.

This is called the soundness condition.

A common condition is the global trace condition: for every infinite path we can eventually trace an object that makes progress infinitely often.

Example. For arithmetic:

- trace objects: variables,
- progress: passing through a case distinction.

In general, checking the global trace condition is PSPACE-complete.

Overview

To reconcile cyclic proofs with traditional proof theory, we want to transform them into into proofs by induction:

- So far, this has been done for various specific proof systems[1].
- We show how to do this in a general abstract way.
- In addition, we want to preserve the structure of the proof.

In particular:

- We start with an abstract cyclic proof system $\mathcal{C}$.
- We define an induced (non-cyclic) proof system $\mathcal{C}_{\text{ind}}$ that extends the rules of $\mathcal{C}$ with an intuitionistic induction principle.
- We show how any cyclic proof π in $\mathcal{C}$ can be transformed into a finite proof π_{ind} in $\mathcal{C}_{\text{ind}}$ with the same conclusion and structure.

By the Curry-Howard correspondence, this applies to recursive functions.

Our approach is greatly inspired by the PhD thesis of Dominik Wehr.

[1] arithmetic, modal/linear/first-order μ-calculus, inductive definitions, ...

Size-change graph

Definition (size-change graph)

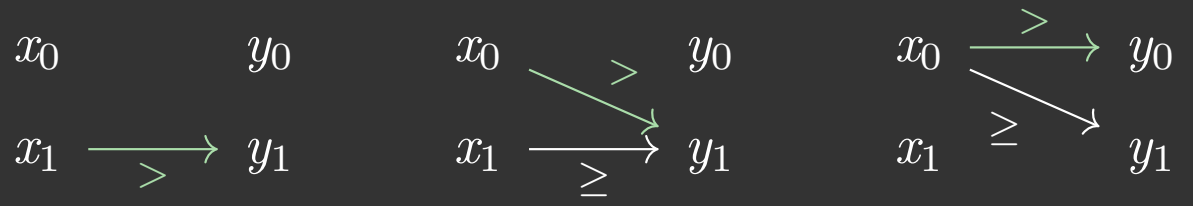
A *size-change graph* $G : X \to Y$ is a bipartite graph from a finite set X to a finite set Y where edges are labelled as either:

$$\geq \text{ (preserving)} \qquad \text{or} \qquad > \text{ (progressing)}.$$

A *trace* through a sequence of size-change graphs $(G_i : X_i \to X_{i+1})_i$ consists of a starting time $k \in \mathbb{N}$ and nodes $(x_i \in X_i)_{i \geq k}$ such that x_i and x_{i+1} are connected by an edge in G_i.

We call an infinite sequence *progressing* if there exists a trace where the connecting edge is progressing infinitely often.

Examples:



Cyclic proof system

Definition (cyclic proof system)

A *proof system* consists of:

- a set Judg *(the judgments)*, a set Rule *(the rule instances)*,
- for each rule $r \in \mathrm{Rule}$ an associated finite list of *premises* $\mathrm{Prem}(r) \in \mathrm{Judg}^*$ and a *conclusion* $\mathrm{concl}(r) \in \mathrm{Judg}$.

A *cyclic proof system* is a proof system with additional structure:

- for each judgment $A \in \mathrm{Judg}$ a number of *(trace) objects* $\mathrm{ob}(A) \in \mathbb{N}$,
- for each rule $r \in \mathrm{Rul}$ and each premise $\mathrm{Prem}(r)_i$ a *size-change graph*

$$\mathrm{Graph}(r)_i : \mathrm{ob}(\mathrm{concl}(r)) \to \mathrm{ob}(\mathrm{Prem}(r)_i).$$

A *derivation* is a (possibly infinite) tree where every node is labelled by a rule in a compatible way. A derivation is called a *proof* if it is *regular* (it only has finitely many subderivations) and for every infinite branch, the induced sequence of size-change graphs is progressing.

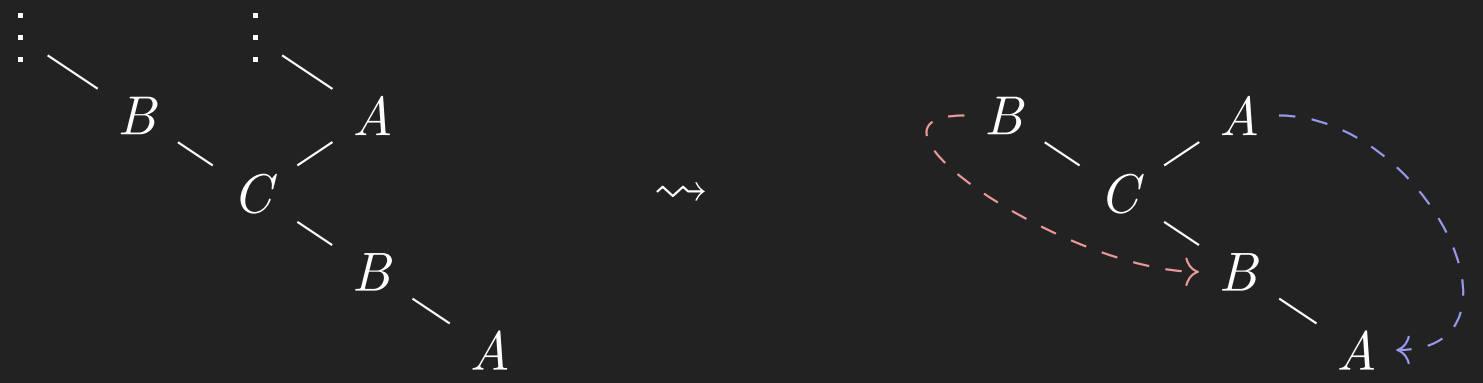
Cyclic representation

So, the proofs are possibly infinite trees. However, because they are regular we can always represent them in a finite way:

- As a finite derivation with back-edges. For a back-edge, we call the starting leaf the bud and the ending internal node the sprout.

We say that the infinite proof has a cyclic representation.

Example. The following infinite proof has a cyclic representation:

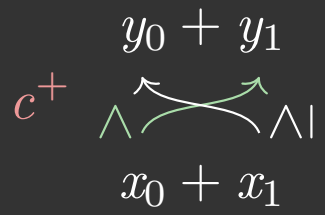


Recursive functions

Proof assistants only allow total functions, an admissive check is the size-change termination condition: for every infinite sequence of recursive calls, we can eventually trace an input that decreases infinite often.

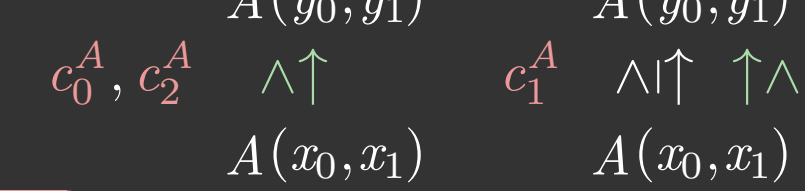
Consider addition with a swapped recursive call:

$$\begin{aligned} 0 + x_1 &:= x_1, \\ \mathsf{suc}(x_0') + x_1 &:= \mathsf{suc}(\overbrace{x_1 + x_0'}^{c^+}), \end{aligned}$$



Or the Ackermann function:

$$\begin{aligned} A(0, x_1) &:= \mathsf{suc}(x_1), \\ A(\mathsf{suc}(x_0'), 0) &:= \underbrace{A(x_0', 1)}_{c_0^A}, \\ A(\mathsf{suc}(x_0'), \mathsf{suc}(x_1')) &:= \underbrace{A(x_0', \overbrace{A(\mathsf{suc}(x_0'), x_1')}^{c_1^A})}_{c_2^A}. \end{aligned}$$



Upshot: much easier to write and read than recursive functions defined via the primitive elimination rules of dependent type theory.

Correspondence

Every set of mutually recursive functions induces a cyclic proof system:

- for every function f we have a judgment $\downarrow f$ (read: f *terminates*) with $\mathsf{ob}(\downarrow f) = \mathsf{ar}(f)$ and a rule

$$\frac{\downarrow g \qquad \text{for every } g \text{ called within } f}{\downarrow f}\, f,$$

  where for every recursive call the size-change graph is given by the relation between the input of f and g.

Note that every function has a unique derivation, and the function is size-change terminating iff this derivation is a proof.

This is an instance of the Curry-Howard correspondence:

Cyclic Proof	Recursive Function
Least Fixpoint Formula	Inductive Type
Greatest Fixpoint Formula	Coinductive Type
Cycle	Recursive Function Call
Soundness Condition	Termination Checking

Inductive proof system

Given a cyclic proof system $\mathcal{C}$ we define a (non-cyclic) proof system $\mathcal{C}_{\mathsf{ind}}$ with an explicit induction rules. The judgements of $\mathcal{C}_{\mathsf{ind}}$ are sequents $\gamma_0, \dots, \gamma_{n-1} \vdash \delta$, consisting of formulas given by the grammar:

$$\phi, \psi, \dots \coloneqq A(\overline{x}) \mid x > y \mid x \geq y \mid \phi \to \psi \mid \forall x\, \psi.$$

Here A is a judgment of $\mathcal{C}$, treated as a relation symbol of arity $\mathsf{ob}(A)$.

$\mathcal{C}_{\mathsf{ind}}$ contains the structural rules (including cut), the normal intuitionistic rules for $\to$ and $\forall$, and:

$$\frac{}{\Gamma \vdash x \geq x}\geq_{\mathsf{refl}}, \quad \frac{\Gamma \vdash x \geq y \quad \Gamma \vdash y \geq z}{\Gamma \vdash x \geq z}\geq_{\mathsf{trans}}, \quad \frac{\Gamma \vdash x > y}{\Gamma \vdash x \geq y}\geq_{\mathsf{subsum}},$$

$$\frac{\Gamma \vdash x \geq y \quad \Gamma \vdash y > z}{\Gamma \vdash x > z}>_{\mathsf{extend}_0}, \quad \frac{\Gamma \vdash x > y \quad \Gamma \vdash y \geq z}{\Gamma \vdash x > z}>_{\mathsf{extend}_1},$$

$$x' \text{ fresh } \frac{\Gamma, \forall x' \, (x > x' \to \phi[x'/x]) \vdash \phi}{\Gamma \vdash \forall x\, \phi}>_{\mathsf{ind}_x}.$$

Inductive proof system

Lastly, for every $\mathcal{C}$-rule

$$\frac{A_i \qquad \text{for every } i < n}{A}\, r,$$

with size-change graphs $(G_i : A \to A_i)_{i<n}$, we have a $\mathcal{C}_{\mathsf{ind}}$-rule

$$\overline{y} \text{ fresh } \frac{\Gamma, \overline{x} \gtrsim_{G_i} \overline{y} \vdash A_i(\overline{y}) \qquad \text{for every } i < n}{\Gamma \vdash A(\overline{x})}\, r.$$

Here $\overline{x} \gtrsim_{G_i} \overline{y}$ contains (strict/non-strict) inequalities according to G_i.

Induction on the entire sequent

Instead of

$$x' \text{ fresh } \frac{\Gamma, \forall x' \, (x > x' \to \phi[x'/x]) \vdash \phi}{\Gamma \vdash \forall x \, \phi} >_{\mathsf{ind}_x}$$

we will use the following variation

$$x', \overline{z}' \text{ fresh } \frac{\Gamma, \forall x', \overline{z}' \, (x > x' \to (\Gamma \to \delta)[x', \overline{z}'/x, \overline{z}]) \vdash \delta}{\Gamma \vdash \delta} >_{\mathsf{ind}'_x}$$

These two induction rules are interderivable; one can derive $>_{\mathsf{ind}'_x}$ by applying $>_{\mathsf{ind}_x}$ to the formula representing the entire sequent:

$$\phi := \forall \overline{z} \, (\Gamma \to \delta).$$

General Idea

We will transform a (possibly infinite) $\mathcal{C}$-proof π of A_0 into a finite $\mathcal{C}_{\mathsf{ind}}$-proof π_{ind} of $\vdash A_0(\overline{x}_0)$:

- We first translate π into a (possibly infinite) derivation $\pi^{\mathsf{inf}}_{\mathsf{ind}}$ in $\mathcal{C}_{\mathsf{ind}}$:

$$\dfrac{\dfrac{\dfrac{\vdots}{A_3}}{A_2}\,r_2}{A_1}\,r_1}{A_0}\,r_0 \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdots}{\overline{x_0} \gtrsim_{G_0} \overline{x_1}, \overline{x_1} \gtrsim_{G_1} \overline{x_2}, \overline{x_2} \gtrsim_{G_2} \overline{x_3} \vdash A_3(\overline{x}_3)}\,r_2}{\overline{x_0} \gtrsim_{G_0} \overline{x_1}, \overline{x_1} \gtrsim_{G_1} \overline{x_2} \vdash A_2(\overline{x}_2)}\,r_1}{\overline{x_0} \gtrsim_{G_0} \overline{x_1} \vdash A_1(\overline{x_1})}\,r_0}{\vdash A_0(\overline{x_0})}$$

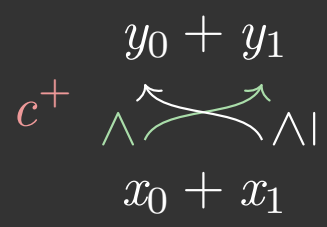
- We will 'cut short' the infinite branches of $\pi^{\mathsf{inf}}_{\mathsf{ind}}$ by introducing induction hypotheses below, and applying them above, yielding π_{ind}.

  For this, we use a nice cyclic representation π_{cyc} of π: we introduce an induction hypothesis at the sprout and apply it at the bud.

Importance of progress

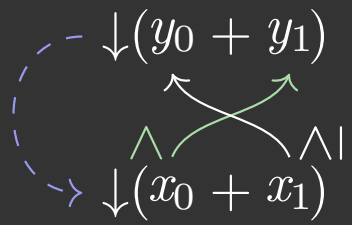
Consider the swapped addition function:

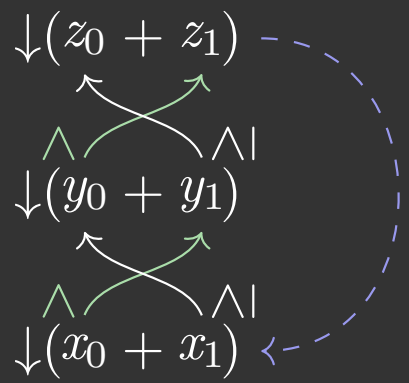
$$0 + x_1 \coloneqq x_1,$$
$$\mathsf{suc}(x_0') + x_1 \coloneqq \mathsf{suc}(\underbrace{x_1 + x_0'}_{c^+}),$$



$$\dfrac{\dfrac{\vdots}{\downarrow +}}{\downarrow +} +.$$

The proof π has (among others) the following two cyclic representations, where we have visualized the size-change graphs:





The second representation can be used to prune the infinite derivation $\pi_{\mathsf{ind}}^{\mathsf{inf}}$ into a finite proof π_{ind} (since there is an input with progress).

Importance of progress

The infinite derivation $\pi_{\mathsf{ind}}^{\mathsf{inf}}$ is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{x_0 > y_1, x_1 \geq y_0, y_0 > z_1, y_1 \geq z_0 \vdash \downarrow(z_0 + z_1)}+}{x_0 > y_1, x_1 \geq y_0 \vdash \downarrow(y_0 + y_1)}+}{\vdash \downarrow(x_0 + x_1)}+$$

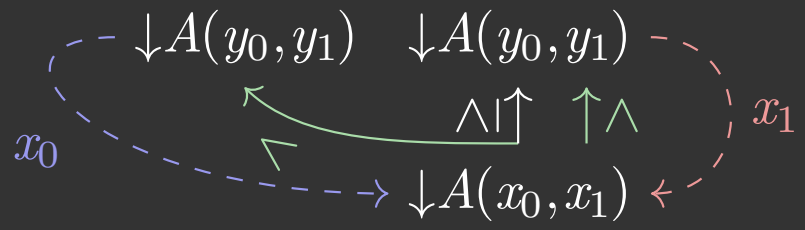
The cyclic representation of the previous slide yields π_{ind}:

$$\dfrac{\dfrac{\dfrac{\dfrac{(\text{use } \mathsf{hyp}(x_0) \text{ on } x_0' := z_0 \text{ and } x_1' := z_1)}{\mathsf{hyp}(x_0), x_0 > y_1, x_1 \geq y_0, y_0 > z_1, y_1 \geq z_0 \vdash \downarrow(z_0 + z_1)}+}{\mathsf{hyp}(x_0), x_0 > y_1, x_1 \geq y_0 \vdash \downarrow(y_0 + y_1)}+}{\mathsf{hyp}(x_0) \vdash \downarrow(x_0 + x_1)} >_{\mathsf{ind}'_{x_0}},$$

where $\mathsf{hyp}(x_0) := \forall x_0' x_1' \, (x_0 > x_0' \to \downarrow(x_0' + x_1'))$.

Importance of induction order

The termination proof of the Ackermann function has cyclic repr:



We introduce an induction hypothesis for x_0 and x_1 in order:

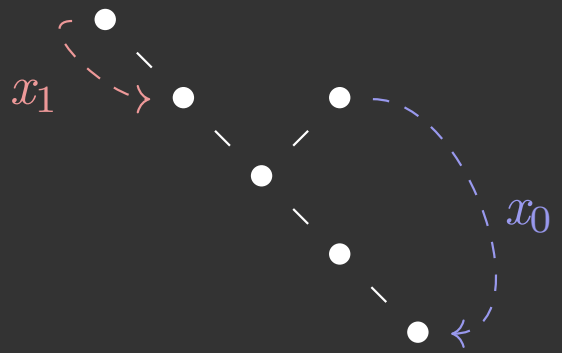
$$\dfrac{\dfrac{\dfrac{\dfrac{(\text{use } \mathsf{hyp}_0(x_0) \text{ on } x_i' := y_i)}{\ldots, x_0 > y_0 \vdash \downarrow A(y_0, y_1)} \quad \dfrac{(\text{use } \mathsf{hyp}_1(x_1) \text{ on } x_i' := y_i)}{\ldots, x_0 \geq y_0, x_1 > y_1 \vdash \downarrow A(y_0, y_1)}}{\mathsf{hyp}_0(x_0), \mathsf{hyp}_1(x_1) \vdash \downarrow A(x_0, x_1)}\, A}{\mathsf{hyp}_0(x_0) \vdash \downarrow A(x_0, x_1)}\, >_{\mathsf{ind}'_{x_1}}}{\vdash \downarrow A(x_0, x_1)}\, >_{\mathsf{ind}'_{x_0}},$$

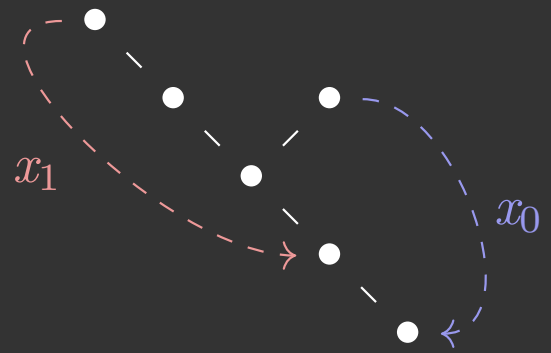
where the induction hypotheses are:

$$\mathsf{hyp}_0(x_0) := \forall x_0' \, \forall x_1' \, (x_0 > x_0' \to \downarrow A(x_0', x_1')),$$
$$\mathsf{hyp}_1(x_1) := \forall x_0' \, \forall x_1' \, (x_1 > x_1' \to \mathsf{hyp}_0(x_0') \to \downarrow A(x_0', x_1')).$$

Importance of forgetting

Consider the following two forms of cyclic representations:





For both, if we want to transform them into finite $\mathcal{C}_{\text{ind}}$-proofs, we have to introduce an induction hypothesis at the root for x_0:

- in the first example, we can forget the induction hypothesis for x_0 (using weakening) before introducing the induction hypothesis for x_1;
- in the second example this is not possible, so in this example the back-edge for x_1 has to 'preserve' x_0 in some way.

Non-linear

The main technical difficulty is finding a suitable cyclic representation:

- We use annotations based on 'reset proofs' and the 'Safra construction' to find a cyclic representation with an induction order.
- We use a technique of Sprenger and Dam to unfold the tree until the order of sprouts in the tree respects the induction order.

Here we modify reset proofs in one important way:

- Instead doing a reset whenever we have a 'cover', we wait until we have a 'uniform cover'.

This allows us to remove the restriction that the well-founded relation $\leq$ is a linear order: we only assume that it is a quasi-order.

Conclusion

Theorem

For every $\mathcal{C}$-proof π of A, there exists a $\mathcal{C}_{\mathsf{ind}}$-proof π_{ind} of $\vdash A(\overline{x}_0)$. Moreover, π_{ind} preserves the structure of π: the $\mathcal{C}$-rules of π_{ind} form a finite subtree of π.

We also have a multi-sorted version of this theorem: here the trace objects of $\mathcal{C}$ are given sorts, and $\mathcal{C}_{\mathsf{ind}}$ is a multi-sorted first-order logic.

Future Work

- Extend to a mix between inductive and coinductive sorts:
  - Either by allowing both inductive and coinductive traces.
  - Or through ordinal approximations or sized types.
- See whether the results hold in a constructive metatheory.
- Apply our result to dependent type theory: show that size-change termination is conservative over primitive elimination rules (extending the results in the PhD theses of McBride and Cockx):
  - Here we have some ideas for inductive types.
  - We want to extend to cover indexed inductive types.

Preprint

https://arxiv.org/abs/2602.12054

Application: cyclic Heyting/Peano arithmetic

Our results can be used to transform a CHA (cyclic Heyting arithmetic) proof into a normal HA (Heyting arithmetic) proof:

- First we translate the CHA-proof into a $\mathsf{CHA}_{\mathsf{ind}}$-proof.
- Because HA and CHA are already first-order theories containing intuitionistic rules for $\to$ and $\forall$, and we can define

$$(n \leq m) \coloneqq \exists x\,(n + x = m), \qquad (n < m) \coloneqq (\mathsf{suc}(n) \leq m),$$

  we can translate the $\mathsf{CHA}_{\mathsf{ind}}$-proof into a HA-proof.

The same strategy gives a transformation from CPA into PA.